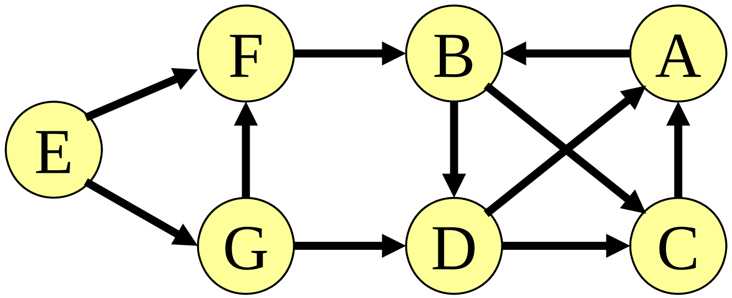
Digraphs: Depth First Search

Given G = (V, E) and all v in V are marked unvisited, a depth-first search (dfs) (generalisation of a pre-order traversal of tree) is one way of navigating through the graph

- **select one v in V and mark as visited**
- **select each unvisited vertex w adjacent to v - dfs(w) (recursive!)**
- **if all vertices marked => search complete**
- **otherwise select an unmarked node and apply dfs**

implementation: adjacency list

DFS: Example

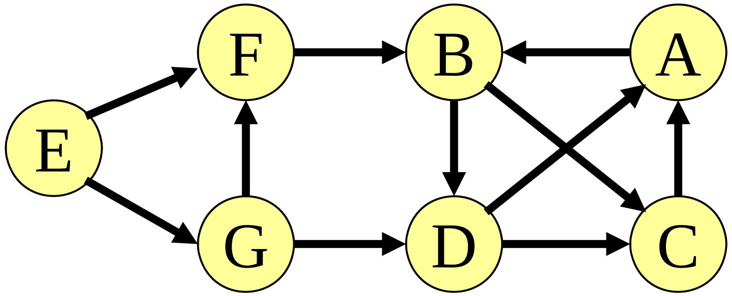


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: A

A, B, C, D, E, F, G

DFS: Example

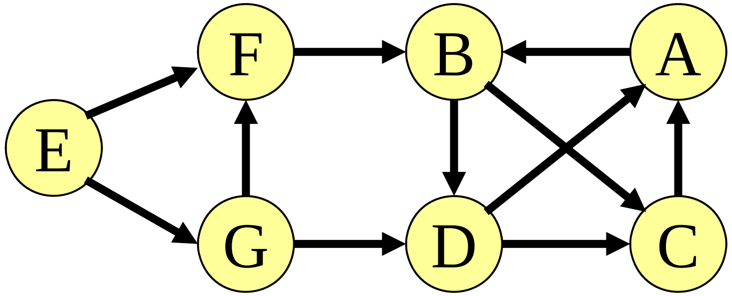


Start: A

A → B

A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

DFS: Example

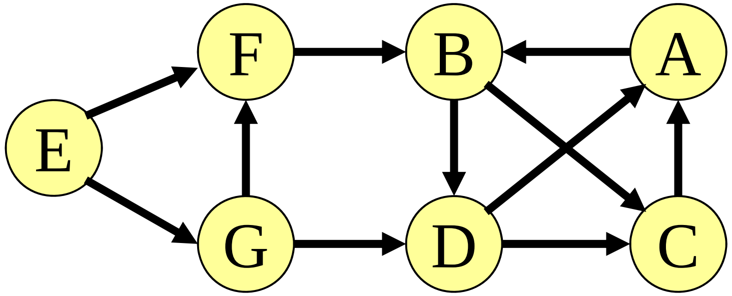


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: A

A → B → C

DFS: Example

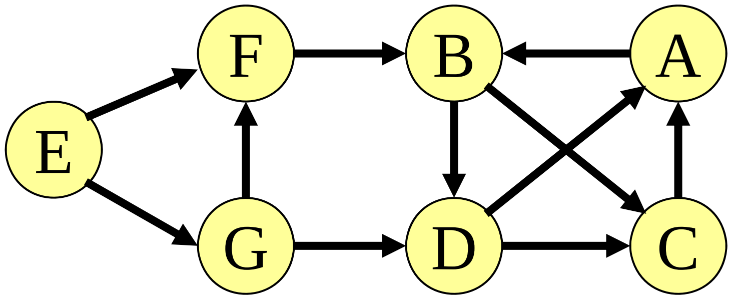


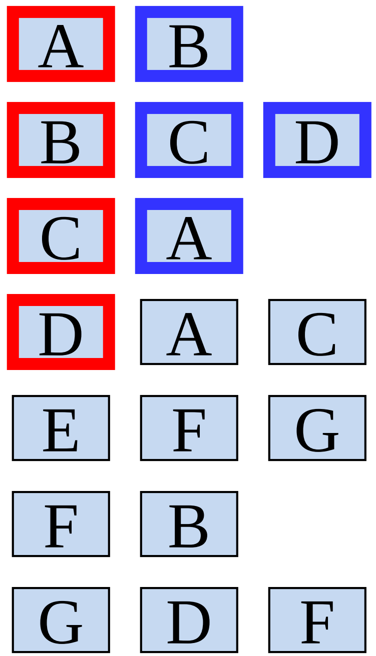
Start: A

A → B → C

A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

DFS: Example

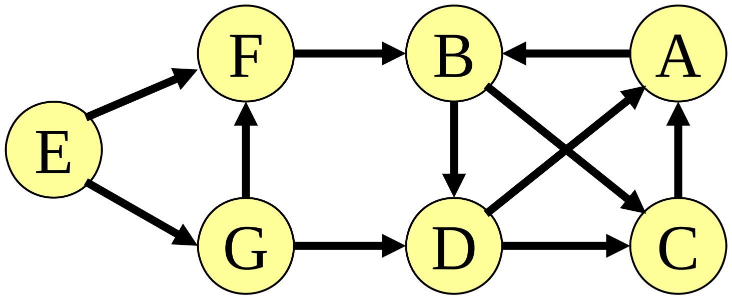




Start: A

A → B → C, B → D

DFS: Example

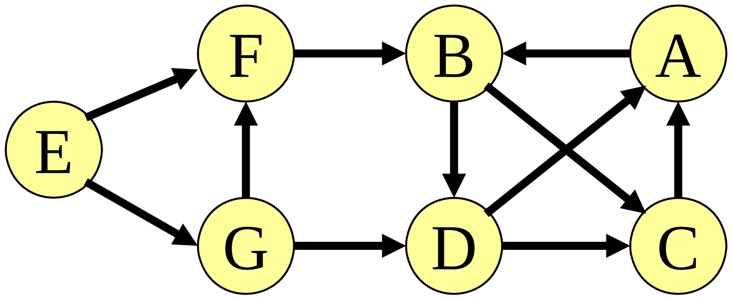


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: A

A → B → C, B → D

DFS: Example



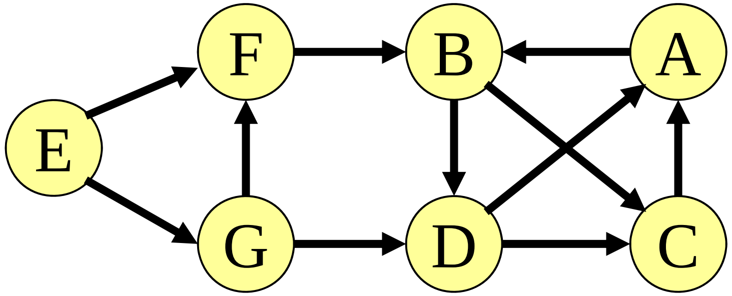
A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

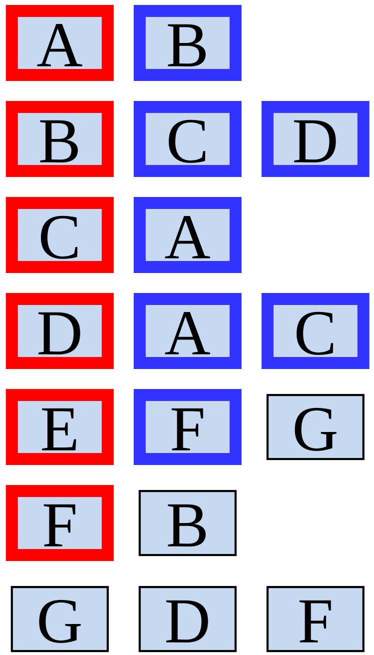
Start: A

A → B → C, B → D

Start E

DFS: Example





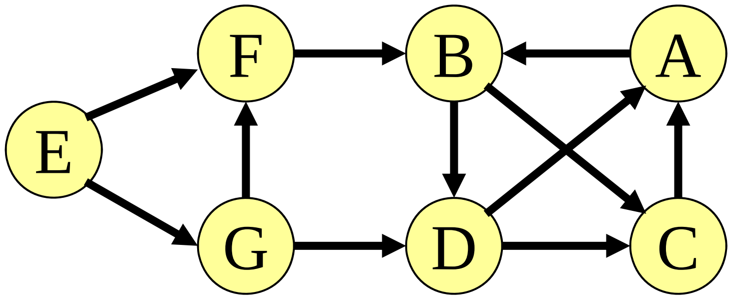
Start: A

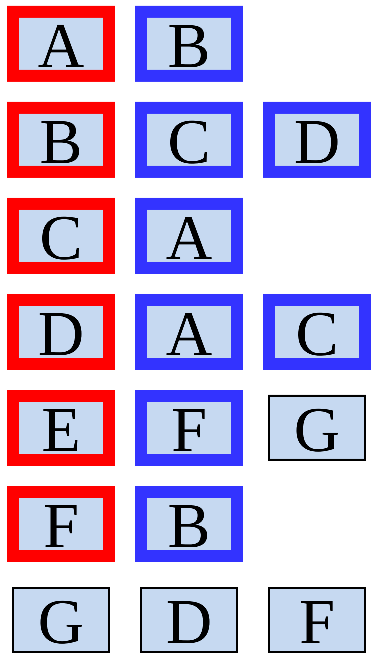
A → B → C, B → D

Start: E

E → F

DFS: Example





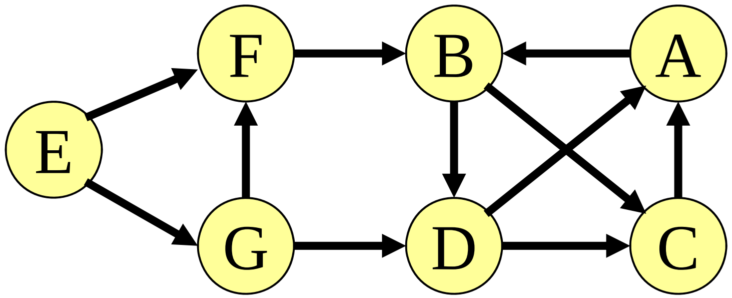
Start: A

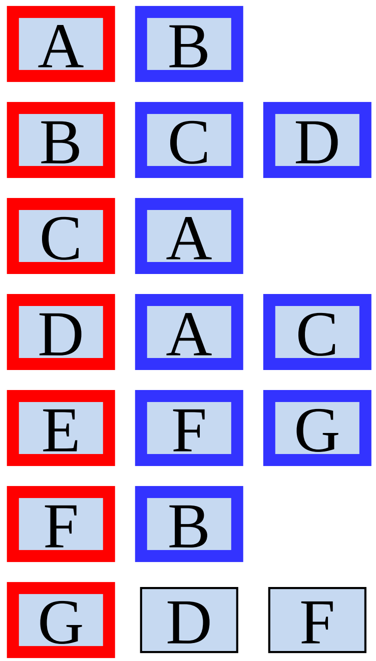
A → B → C, B → D

Start: E

E → F

DFS: Example





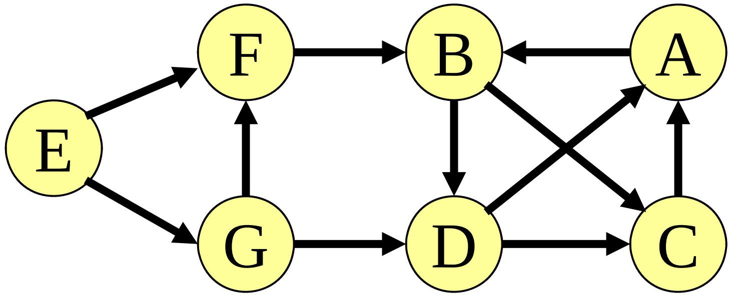
Start: A

A → B → C, B → D

Start: E

E → F, E → G

DFS: Example



A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: A

A → B → C, B → D

Start: E

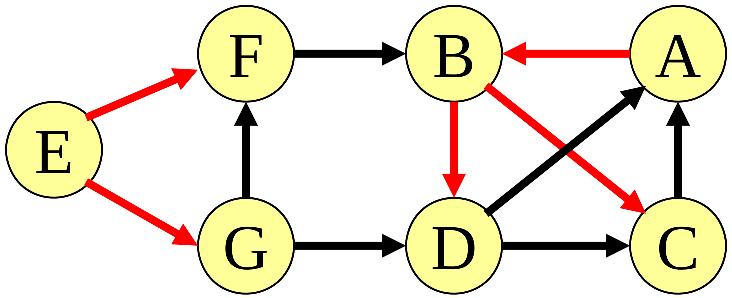
E → F, E → G

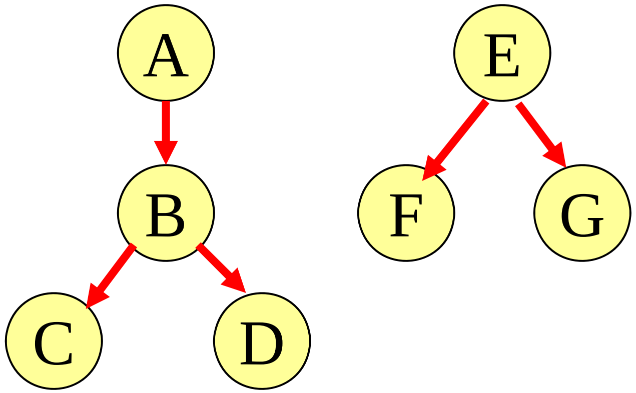
Depth First Spanning Forest

in a dfs of a directed graph, certain edges, when visited, lead to unvisited vertices

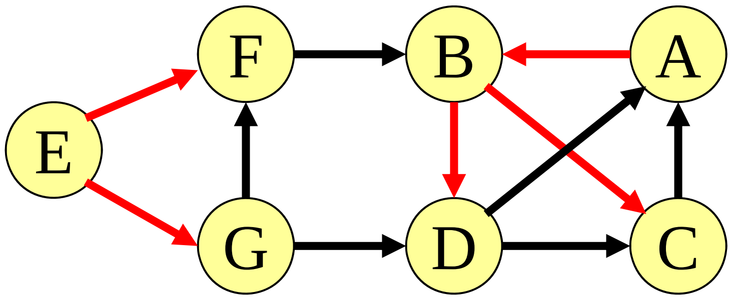
such edges are called TREE EDGES and form a

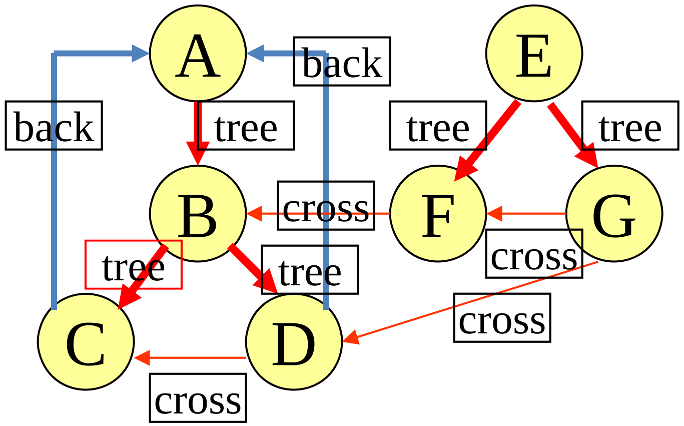
DEPTH FIRST SPANNING FOREST for the given digraph





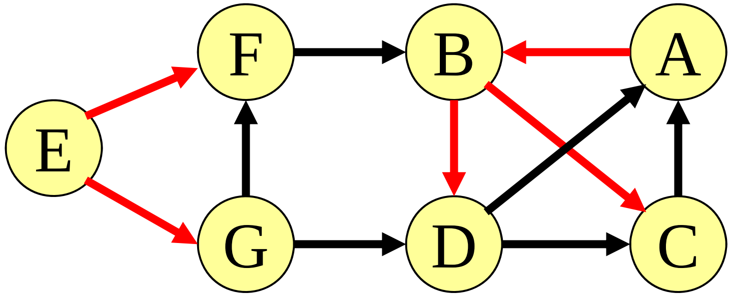
Depth First Spanning Forest

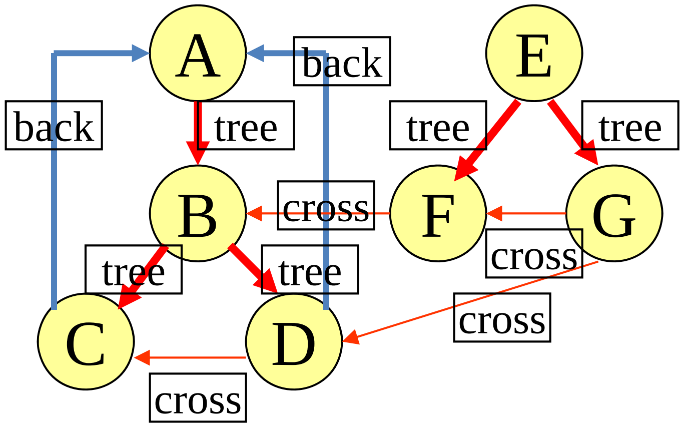




- Other edges are
- back edge
  - vertex to an ancestor
- forward edge
  - non-tree edge from a vertex to a proper descendant (in the tree)
- cross edge
  - edge from V_1 to V_2 - neither an ancestor nor descendant

Depth First Spanning Forest





- Nota Bene (NB)
  - all cross edges go from right to left assuming that
    - children added to tree in order visited (l to r)
    - new trees added to forest in left to right order
- vertices can be numbered (dfn) in depth first order

  A B C D E F G

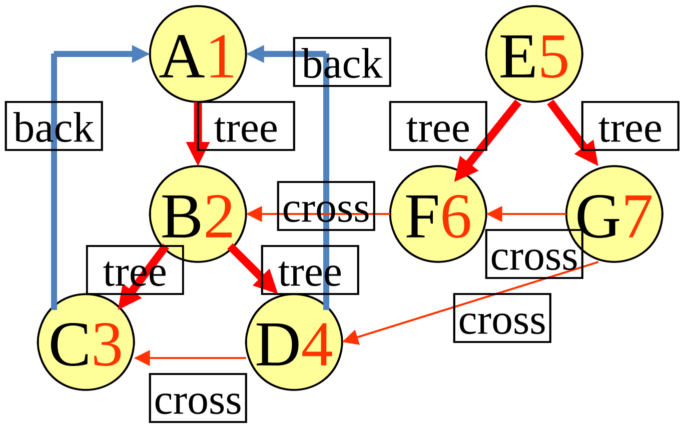
  1 2 3 4 5 6 7

Depth First Spanning Forest

- **All descendants** of v have dfn >= dfn(v)
- **forward edges** low dfn to high dfn
- **back edges** high dfn to low dfn
- **cross edges** high dfn to low dfn

- **back edge => cycle**

- w is a descendant of v iff
  - dfn(v) <= dfn(w) <= dfn(v) + number of descendants of v

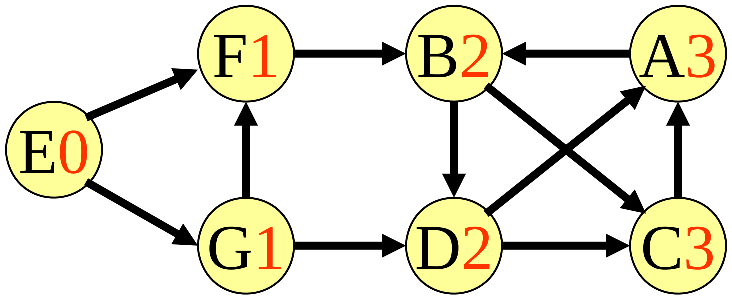


Digraphs: Breadth First Search

Given G = (V, E) and all v in V are marked unvisited, a breadth-first search (bfs) is another way of navigating through the graph

```
select one v in V and mark as visited; enqueue v in Q
while not is_empty(Q) {
    x = front(Q); dequeue(Q);
    for each y in adjacent (x) if unvisited (y) {
            mark(y); enqueue y in Q; process (x,y)
                                        // (e.g. add to tree);
            }
```

BFS: Example

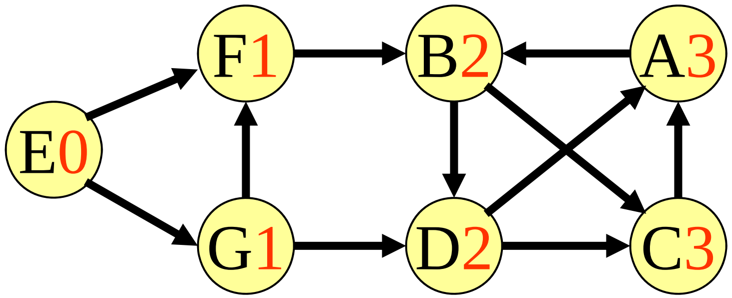


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: E

Output: E, F, G, B, D, C, A

BFS: Example

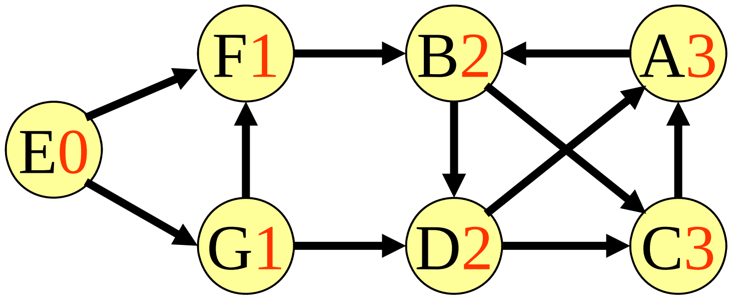


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: E

Q: E

BFS: Example

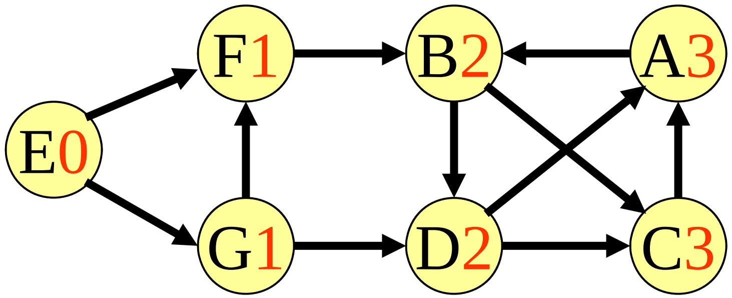


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: E

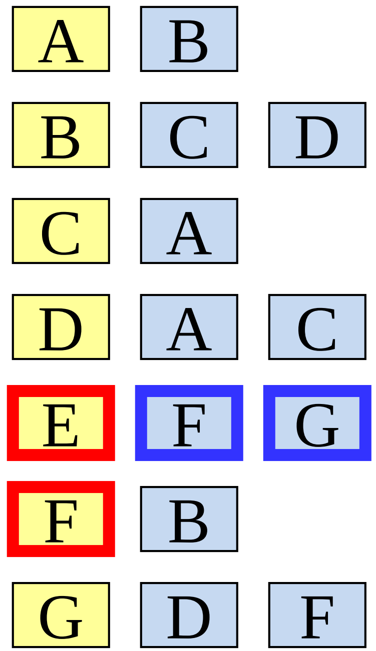
Q: F, G

BFS: Example

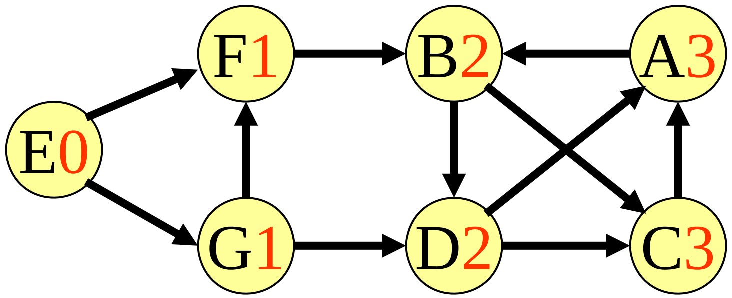


Start: E

Q: G

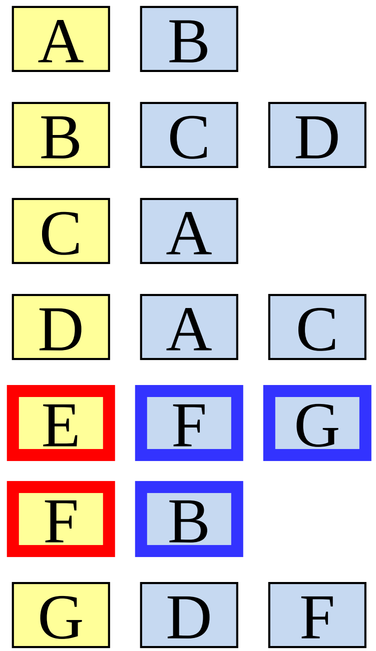


BFS: Example

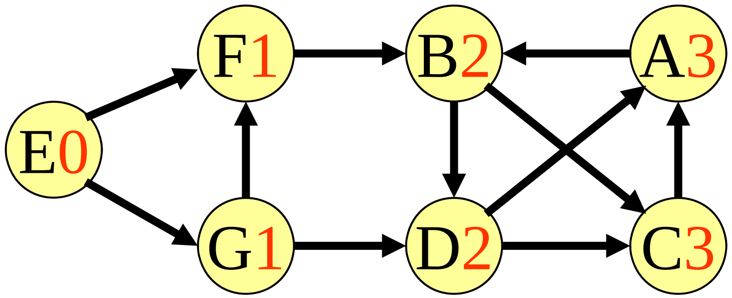


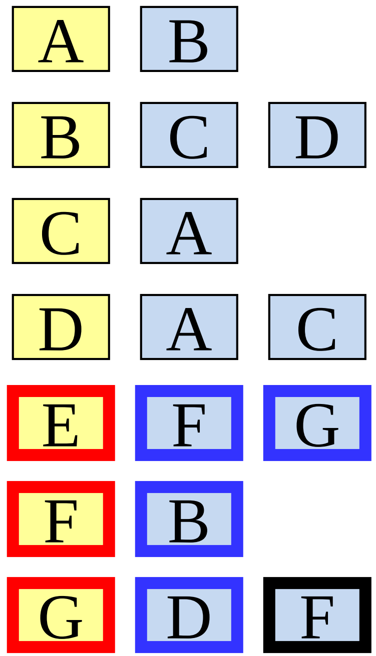
Start: E, F

Q: G, B



BFS: Example

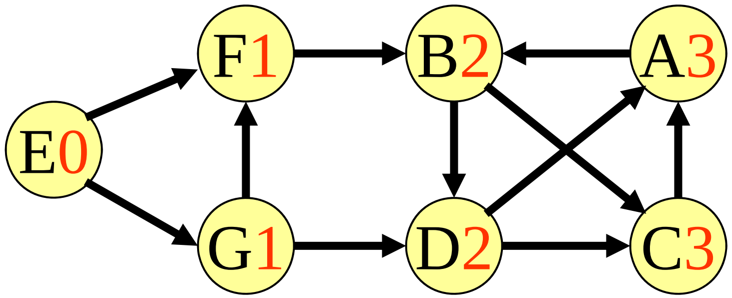




Start: E, F, G

Q: B, D

BFS: Example

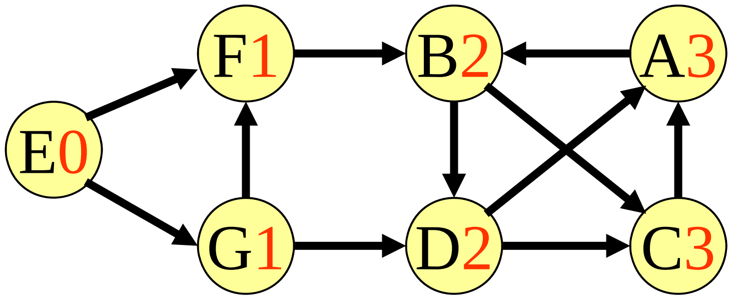


A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: E, F, G, B

Q: D

BFS: Example

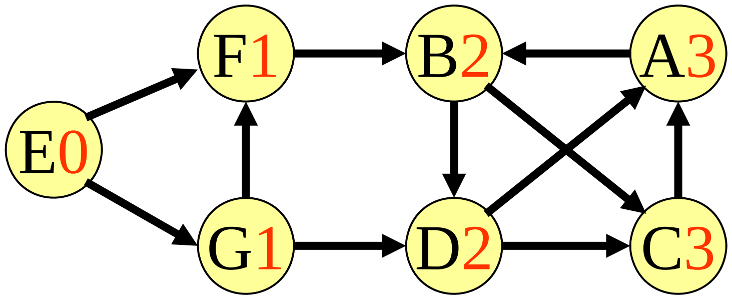


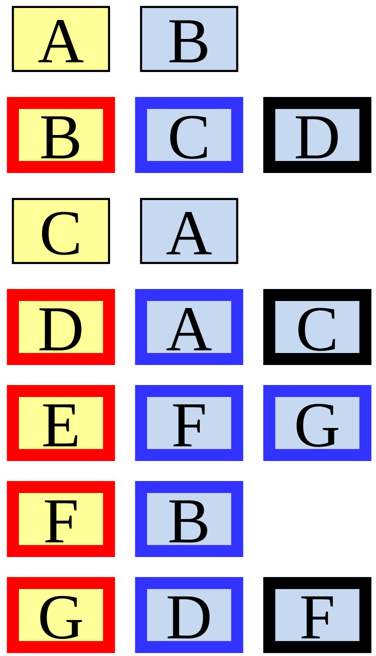
A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

Start: E, F, G, B

Q: D, C

BFS: Example

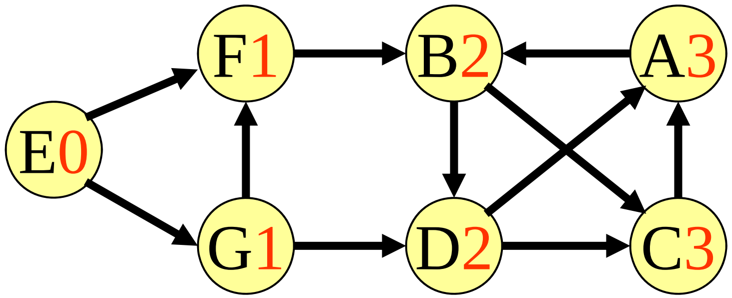


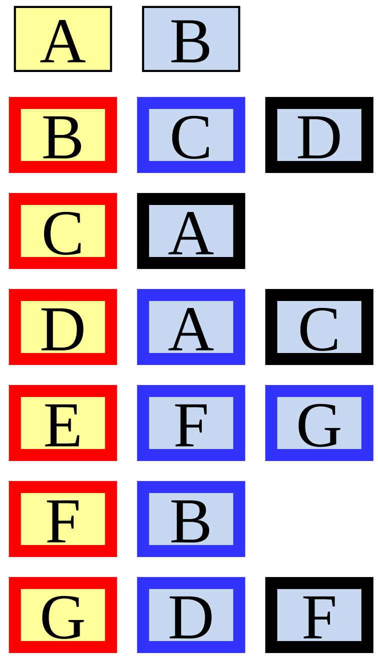


Start: E, F, G, B, D

Q: C, A

BFS: Example

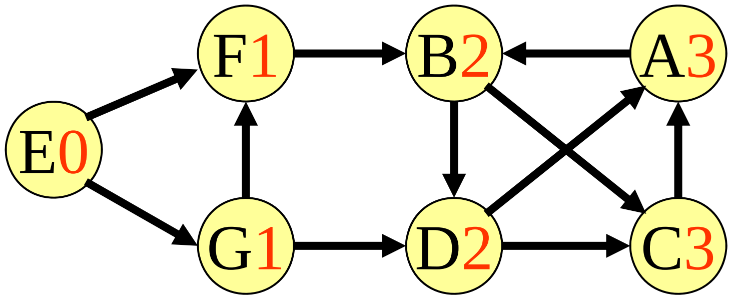




Start: E, F, G, B, D, C

Q: A

BFS: Example



A	B	
B	C	D
C	A	
D	A	C
E	F	G
F	B	
G	D	F

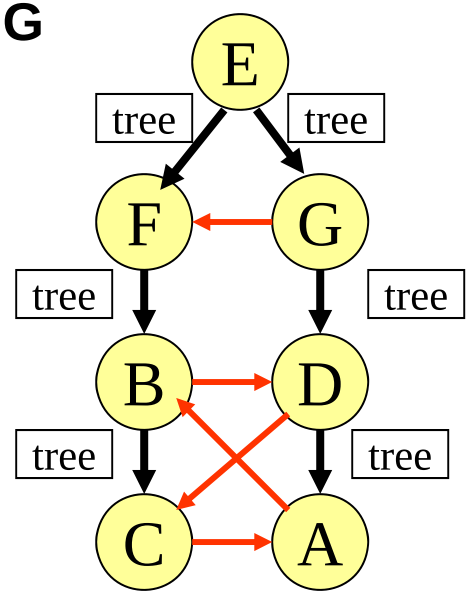
Start: E, F, G, B, D, C, A

Q: empty

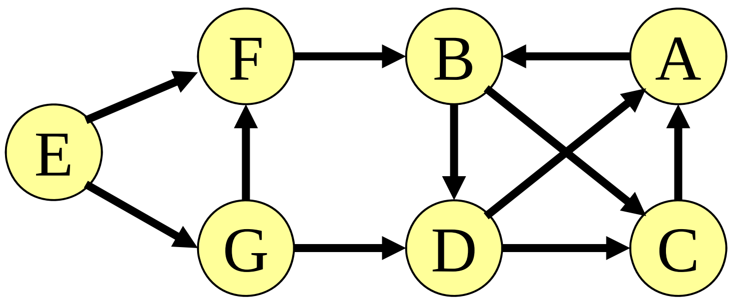
Breadth First Spanning Forest

in a bfs of a directed graph, certain edges, when visited, lead to unvisited vertices- such edges are called TREE EDGES

and form a BREADTH FIRST SPANNING FOREST for the given digraph

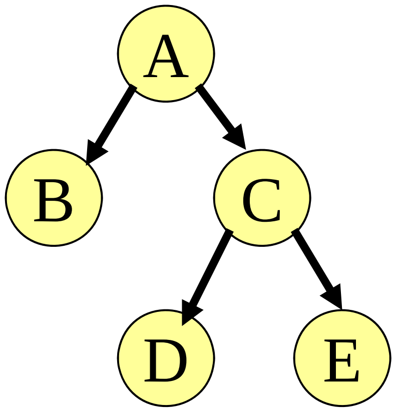


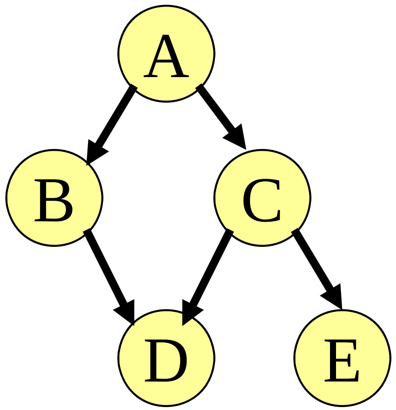
NB only tree & non-tree (cross) edges

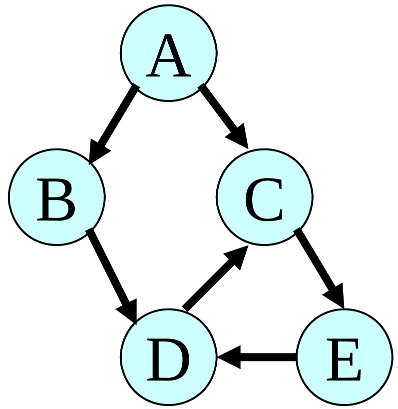


Directed Acyclic Graphs (DAGs)

- DAG - digraph with no cycles
- compare: tree, DAG, digraph with cycle





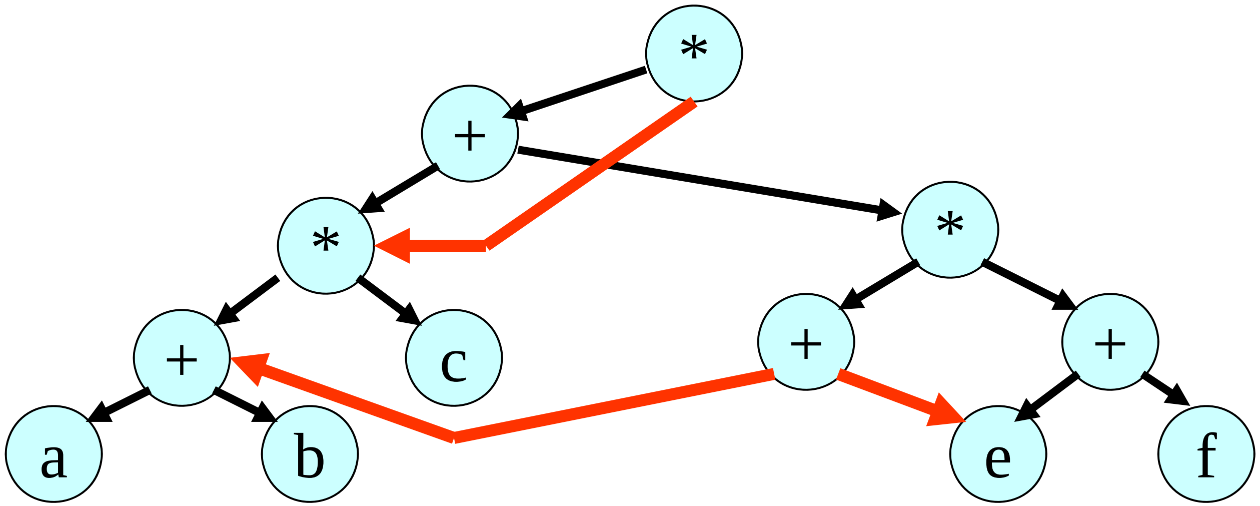


- Tree in-degree = 1 out-degree = 2 (binary)
- DAG in-degree >= 1 out-degree >= 1

DAG: use

- Syntactic structure of arithmetic expressions with common sub-expressions

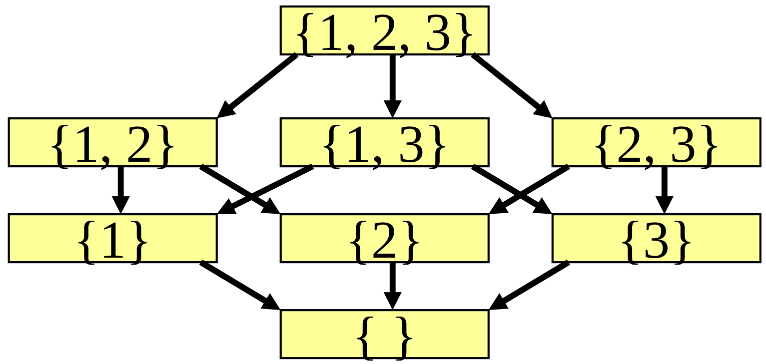
  e.g. ((a+b)*c + ((a+b)+e)*(e+f)) * ((a+b)*c)



DAG: use

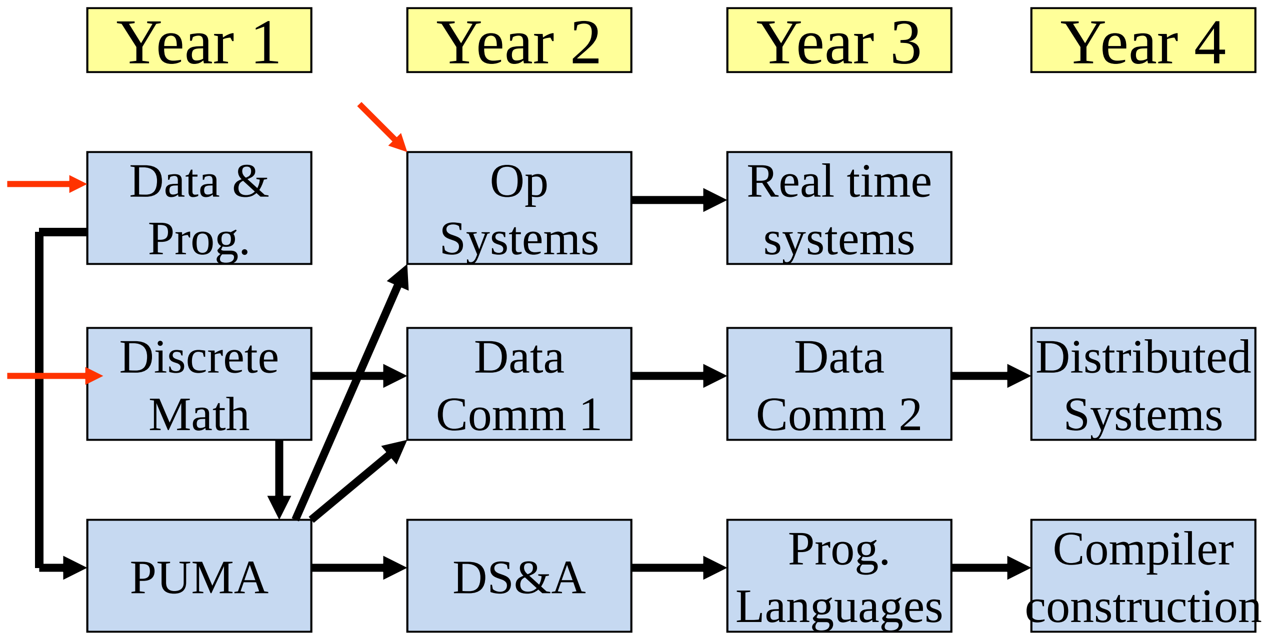
- To represent **partial orders**
- A partial order R on a set S is a binary relation such that
  - for all a in S a R a is false (irreflexive)
  - for all a, b, c in S if a R b and b R c then a R c (transitive)
- examples: “less than” (<) and proper containment on sets

- S ={1, 2, 3}
- P(S) - power set of S (set of all subsets)



DAG: use

- To model course prerequisites or dependent tasks

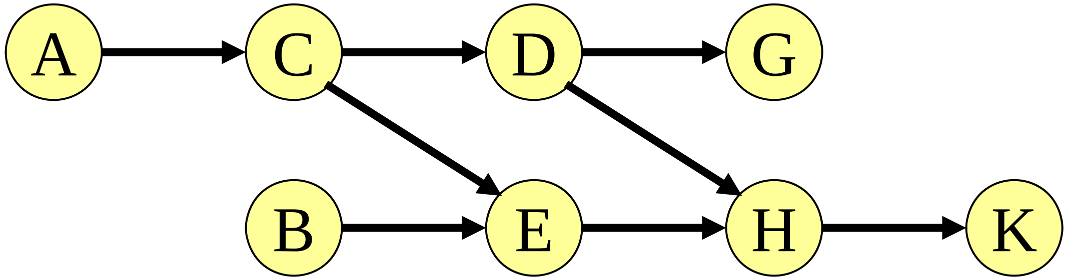


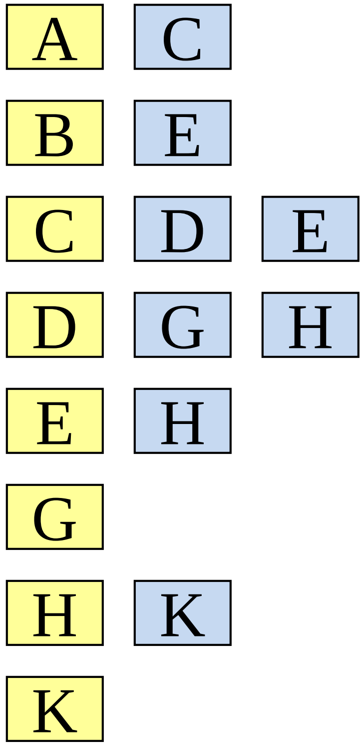
Topological sort

- Given a DAG of prerequisites for courses, a topological sort can be used to determine **<u>an order</u>** in which to take the courses
- (TS: DAG => sequence)   (modified dfs)
- prints **<u>reverse</u>** topological order of a DAG from v

```
tsort(v)    {
   mark v visited
      for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```

Topological sort: example





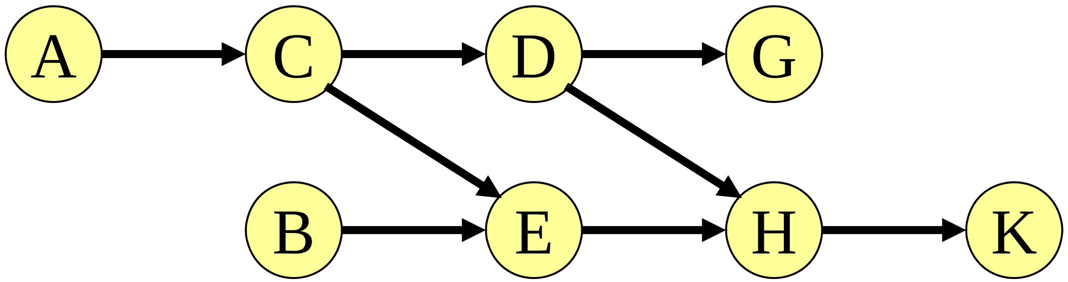
start: A

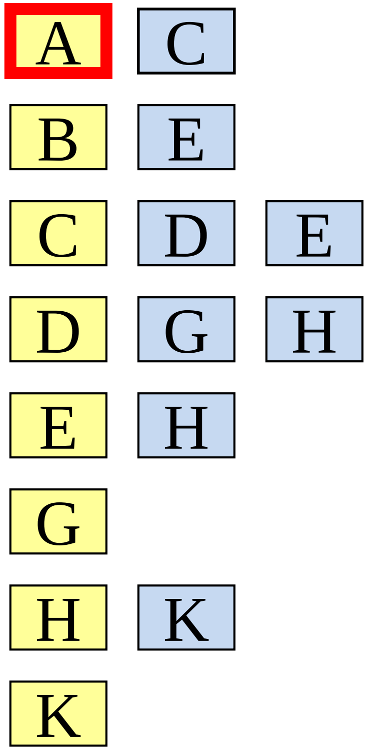
tsort(A) => G K H D E C A B

reverse => B A C E D H K G

Topological Sort example

tsort(v) {
A➔ mark v visited
for each w adjacent to v if w unvisited tsort(w)
display(v)
}





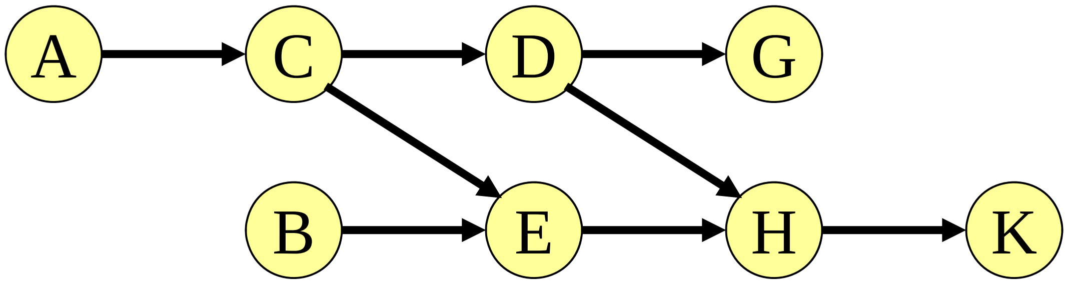
path: **A**

output:

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
  A➔  for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```



A	C	
B	E	
C	D	E
D	G	H
E	H	
G		
H	K	
K		

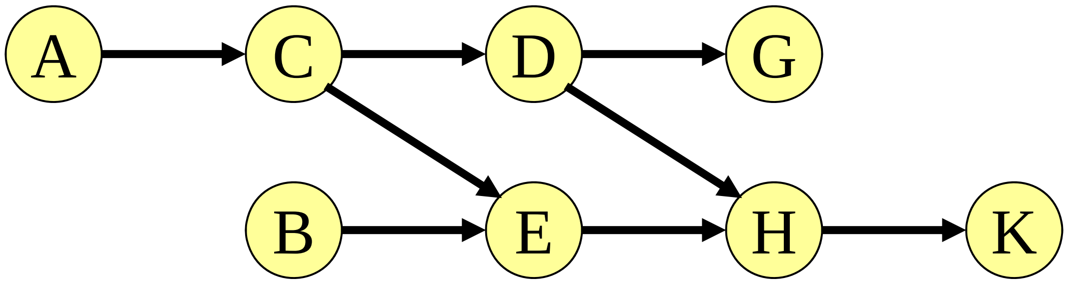
path: **A ➔ C**

output:

reverse:

Topological Sort example

```
tsort(v)    {
 C →mark v visited
      for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```



A	C	
B	E	
C	D	E
D	G	H
E	H	
G		
H	K	
K		

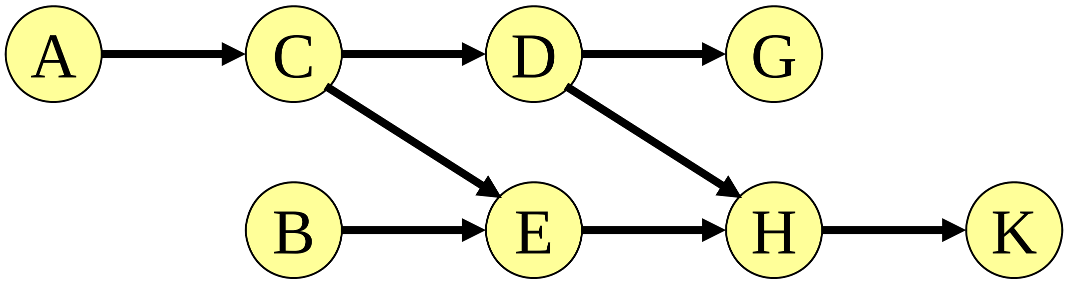
path: **A → C**

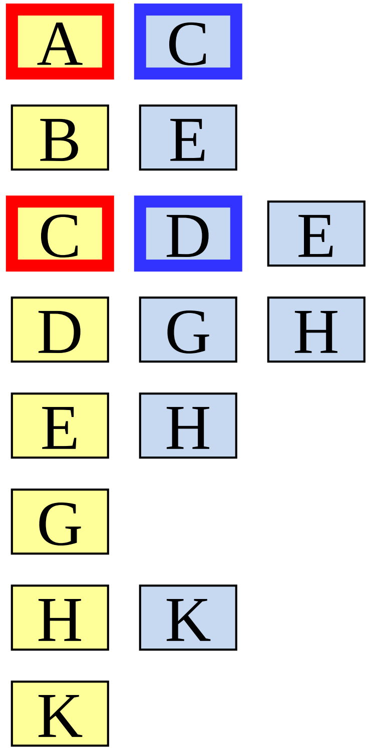
output:

reverse:

Topological Sort example

```
tsort(v)    {
       mark v visited
  C ➔for each w adjacent to v if w unvisited tsort(w)
       display(v)
       }
```





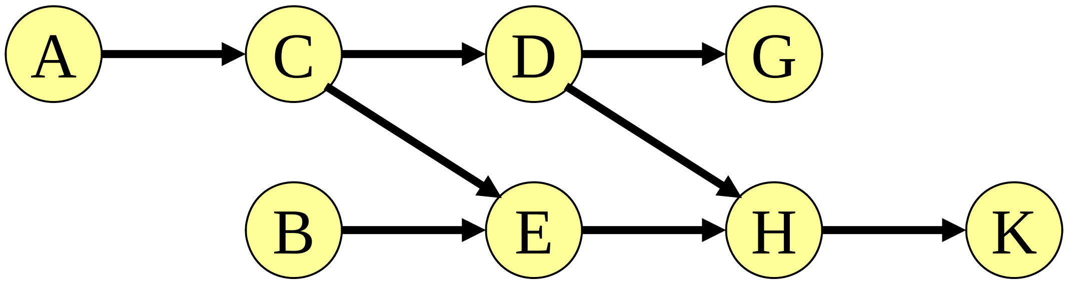
path: **A ➔ C ➔ D**

output:

reverse:

Topological Sort example

```
tsort(v)    {
 D → mark v visited
     for each w adjacent to v if w unvisited tsort(w)
     display(v)
     }
```



A	C	
B	E	
C	D	E
D	G	H
E	H	
G		
H	K	
K		

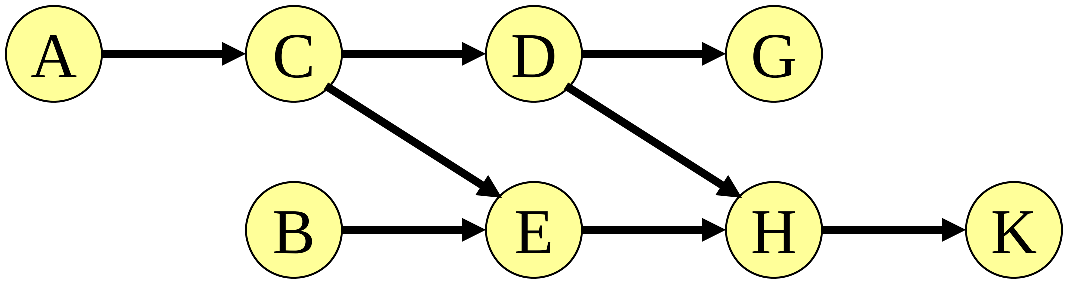
path: **A → C → D**

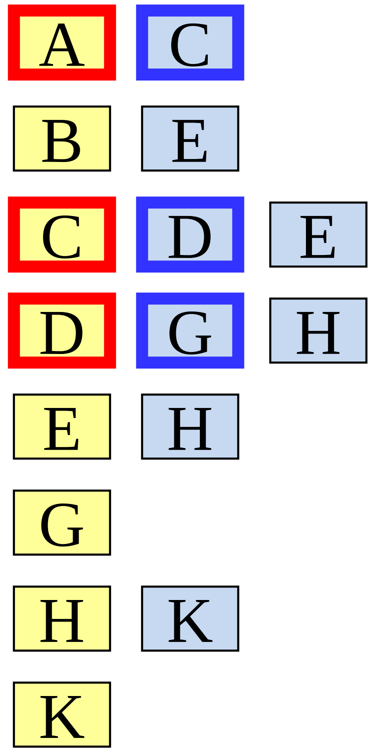
output:

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
  D ➔for each w adjacent to v if w unvisited tsort(w)
        display(v)
        }
```





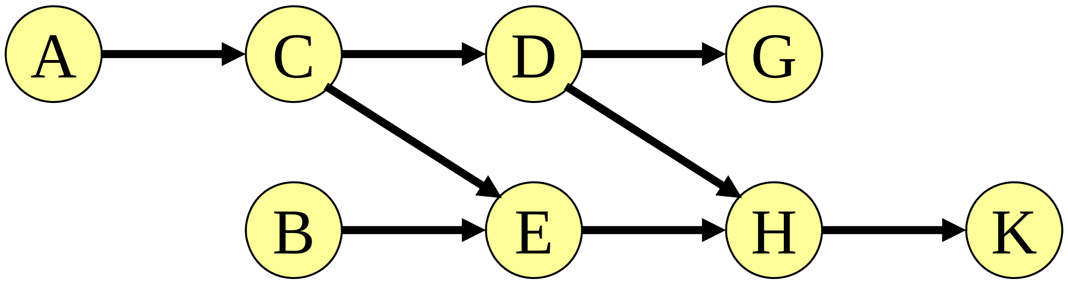
path: **A ➔ C ➔ D ➔ G**

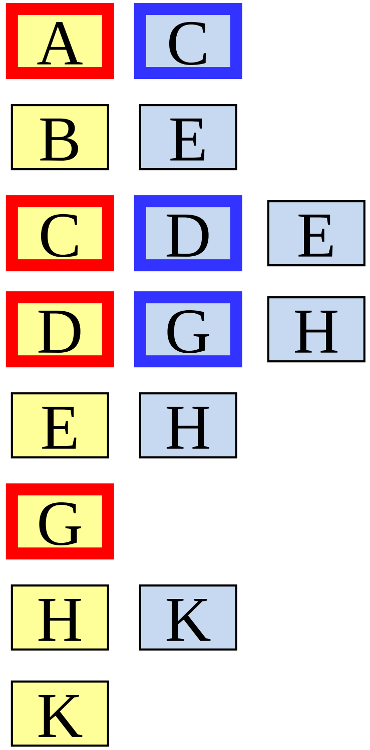
output:

reverse:

Topological Sort example

```
tsort(v)    {
G → mark v visited
      for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





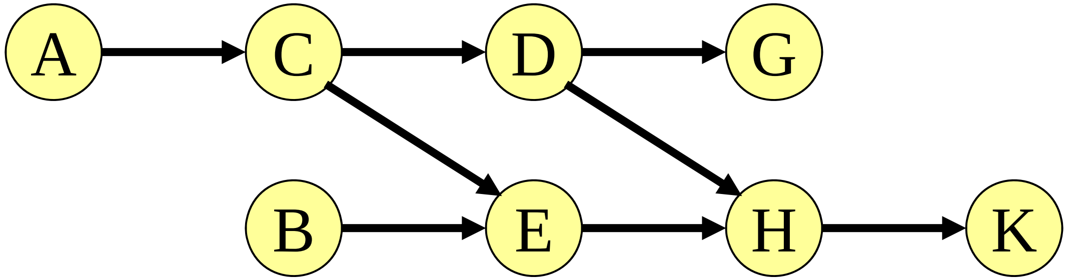
path: **A → C → D → G**

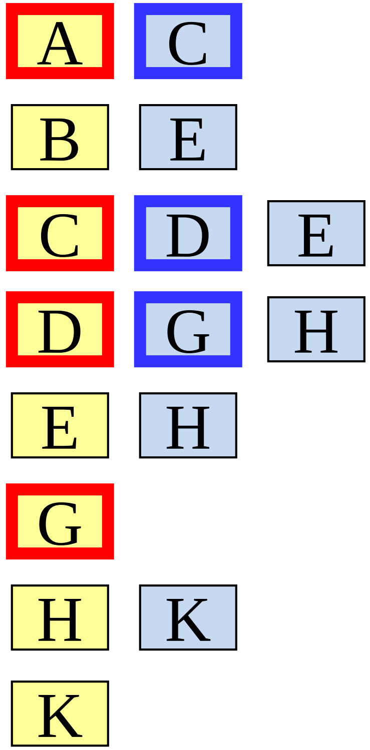
output:

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
 G ➔ for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





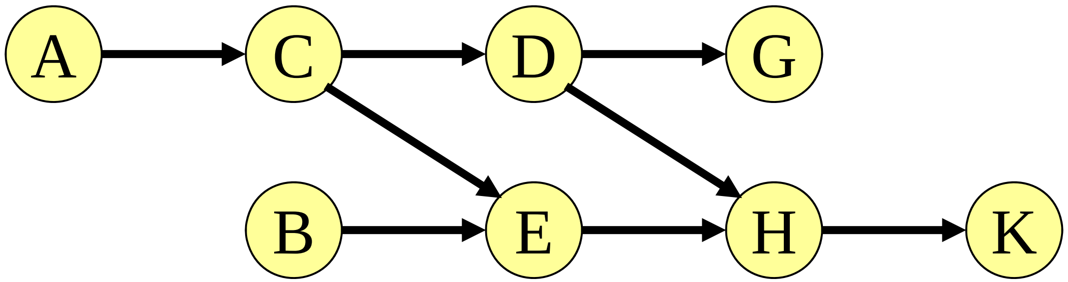
path: **A ➔ C ➔ D ➔ G**

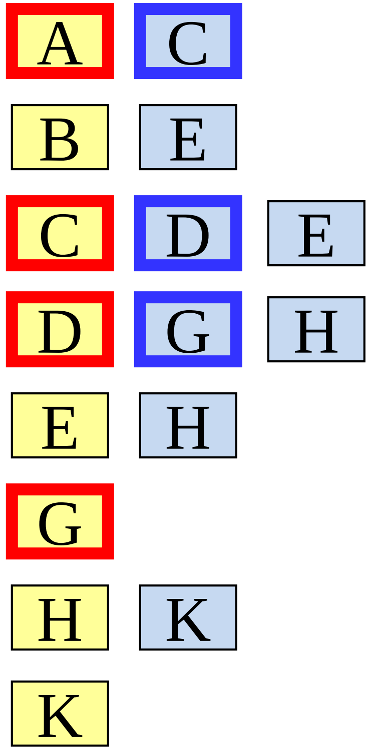
output:

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
      for each w adjacent to v if w unvisited tsort(w)
G → display(v)
      }
```





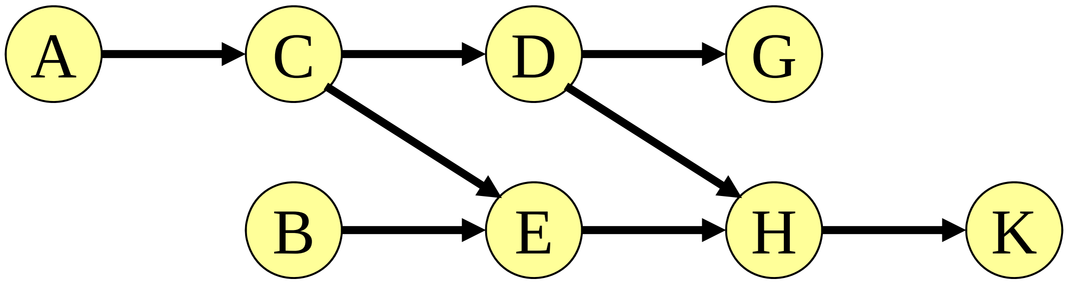
path: **A → C → D → G**

output: **G**

reverse:

Topological Sort example

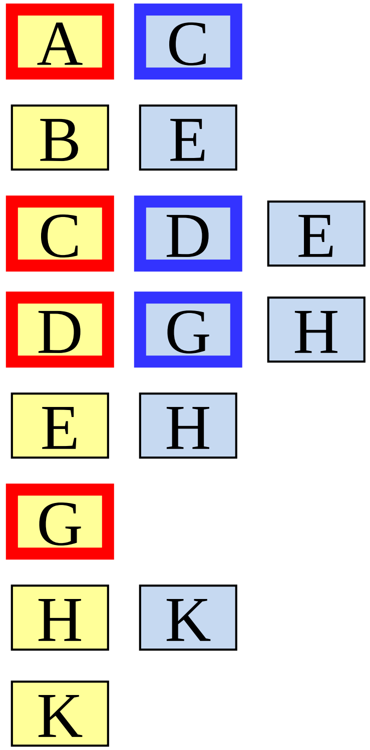
```
tsort(v)    {
      mark v visited
 D ➔ for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```



path: **A ➔ C ➔ D**

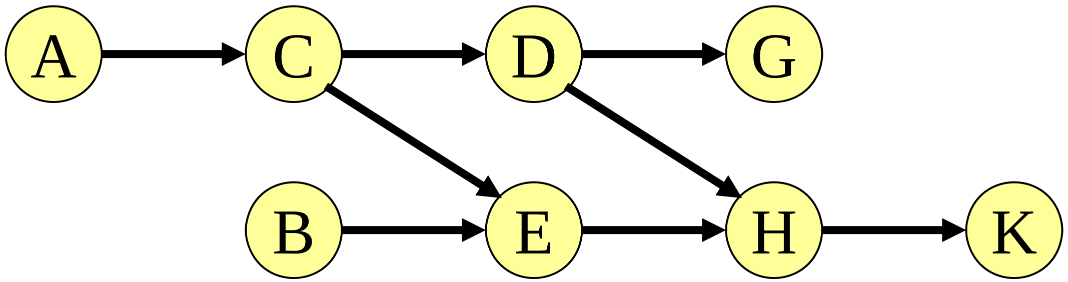
output: **G**

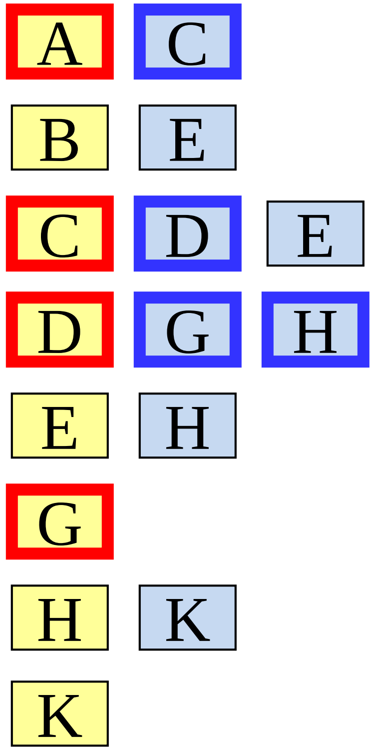
reverse:



Topological Sort example

tsort(v) {
    mark v visited
D ➔ for each w adjacent to v if w unvisited tsort(w)
    display(v)
    }





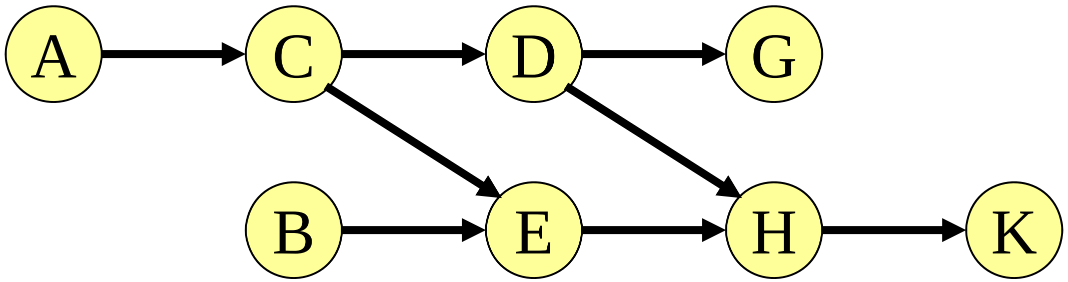
path: A ➔ C ➔ D ➔ H

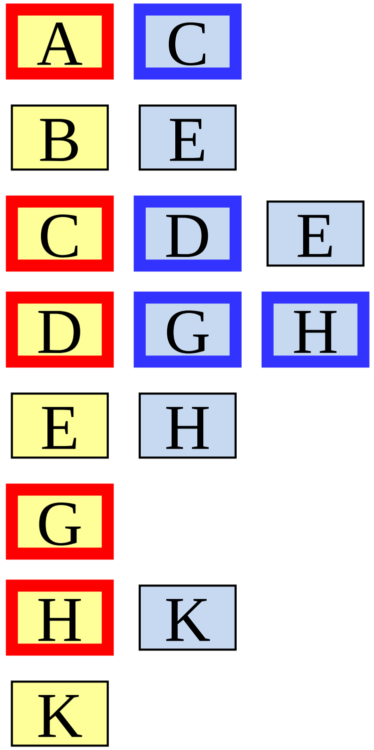
output: G

reverse:

Topological Sort example

tsort(v) {
H ➔ mark v visited
for each w adjacent to v if w unvisited tsort(w)
display(v)
}





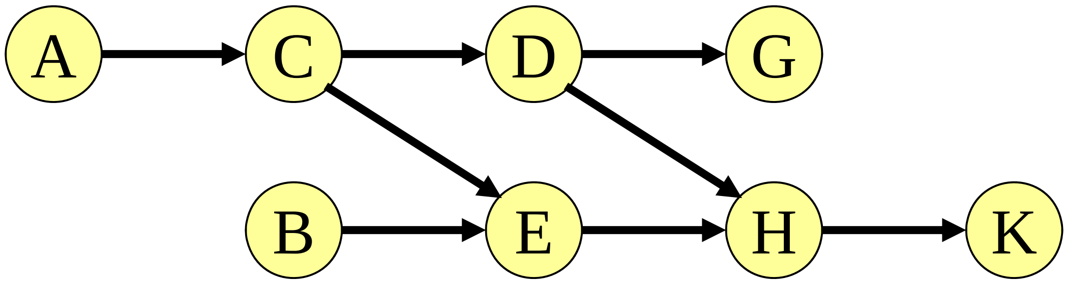
path: A ➔ C ➔ D ➔ H

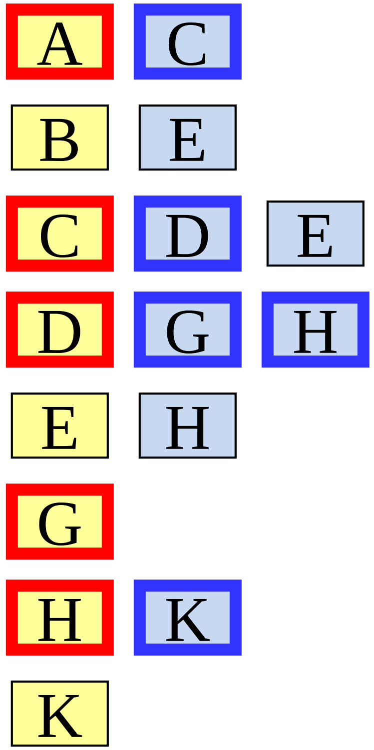
output: G

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
 H ➔  for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





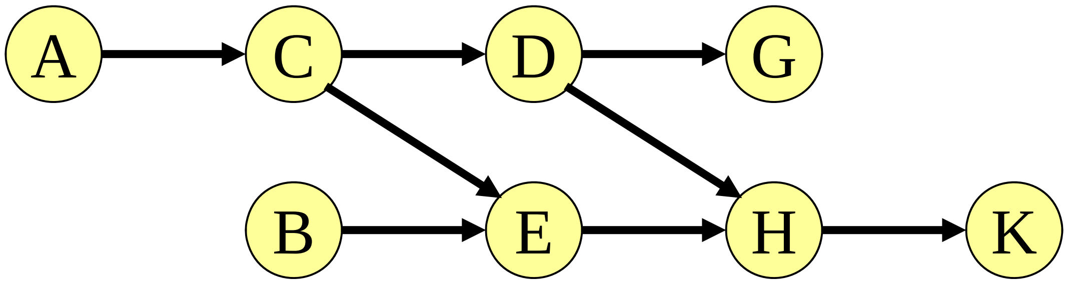
path: A ➔ C ➔ D ➔ H ➔ K

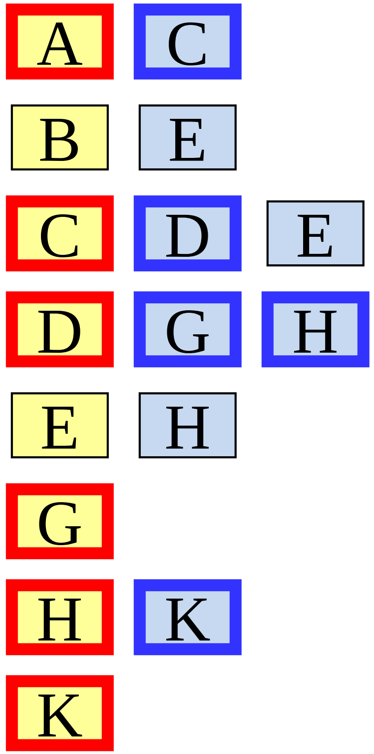
output: G

reverse:

Topological Sort example

tsort(v) {
K ➔ mark v visited
for each w adjacent to v if w unvisited tsort(w)
display(v)
}





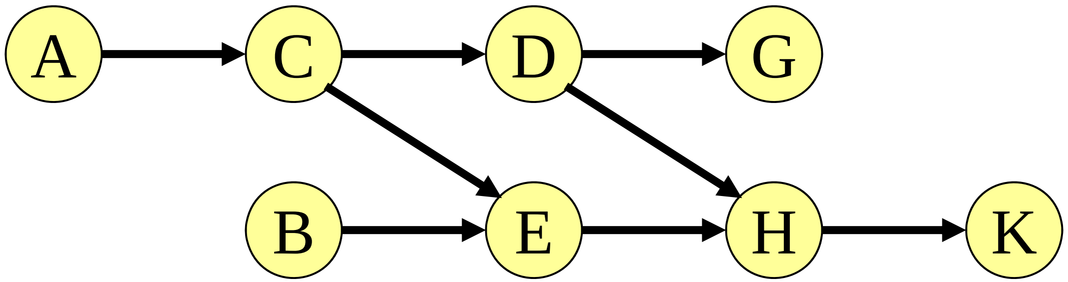
path: A ➔ C ➔ D ➔ H ➔ K

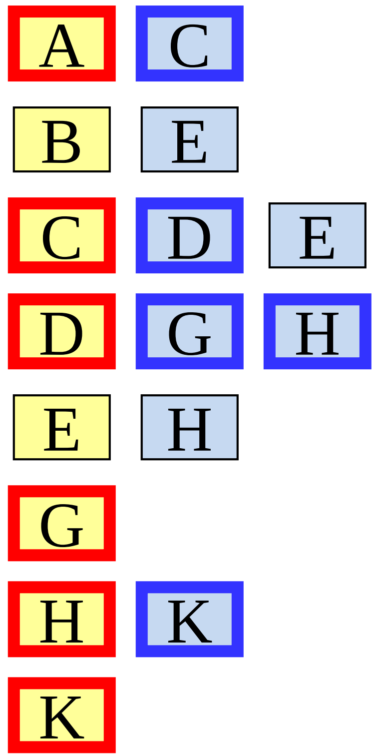
output: G

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
K ➔  for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





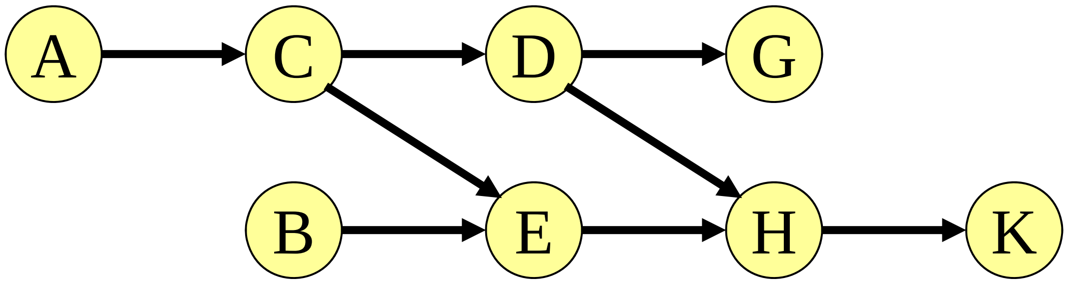
path: **A ➔ C ➔ D ➔ H ➔ K**

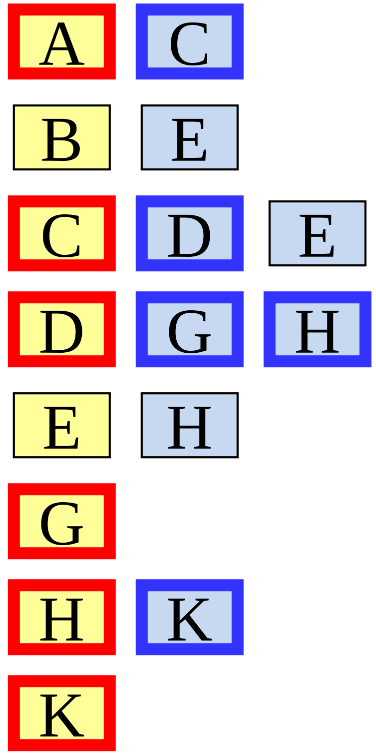
output: **G**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 K ➔ display(v)
        }
```





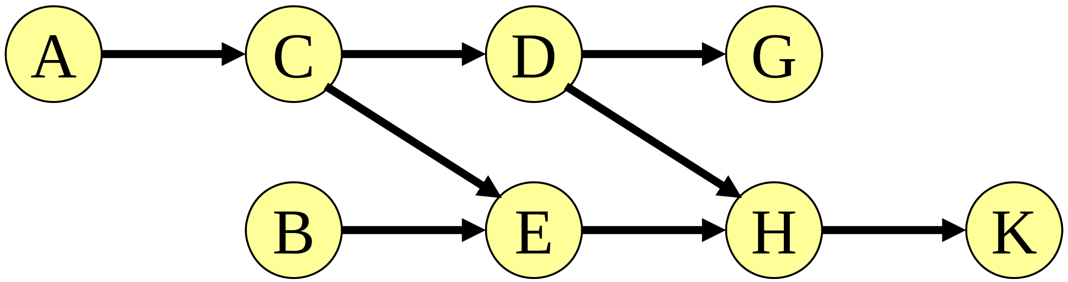
path: A ➔ C ➔ D ➔ H ➔ K

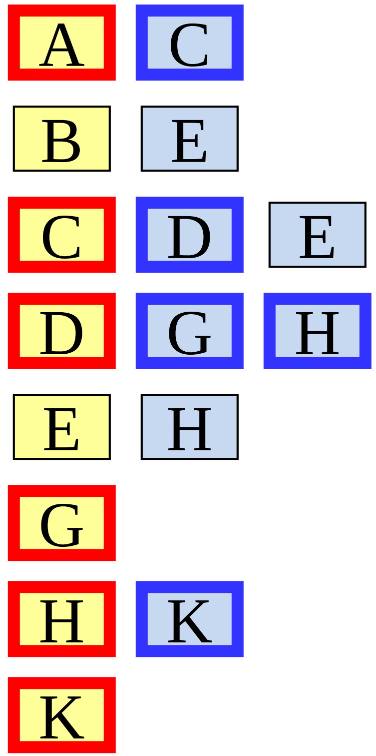
output: G K

reverse:

Topological Sort example

```
tsort(v)     {
      mark v visited
 H ➔  for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





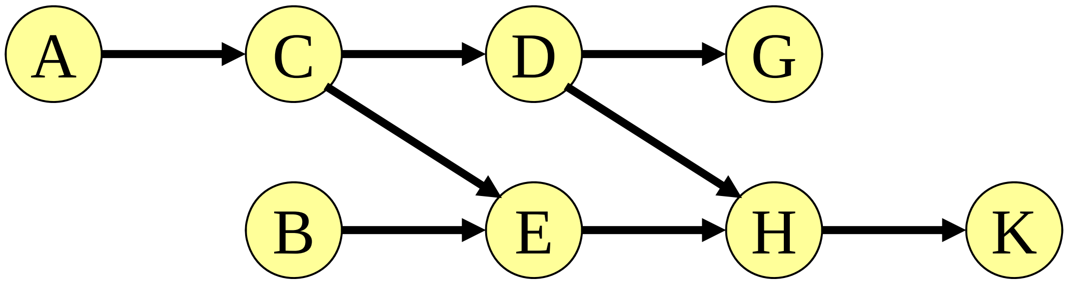
path: **A ➔ C ➔ D ➔ H**

output: **G K**

reverse:

Topological Sort example

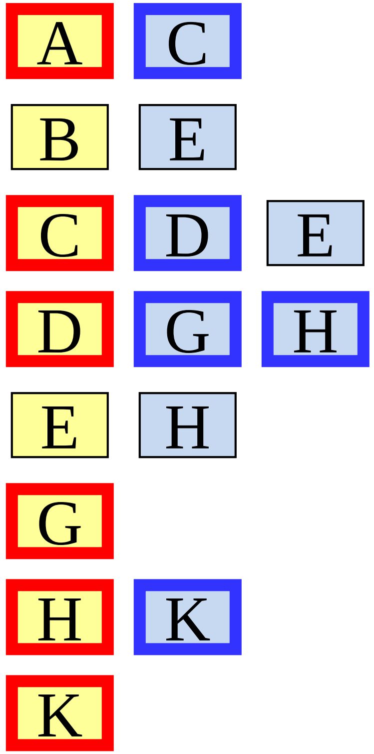
```
tsort(v)    {
      mark v visited
      for each w adjacent to v if w unvisited tsort(w)
 H ➔display(v)
      }
```



path: A ➔ C ➔ D ➔ H

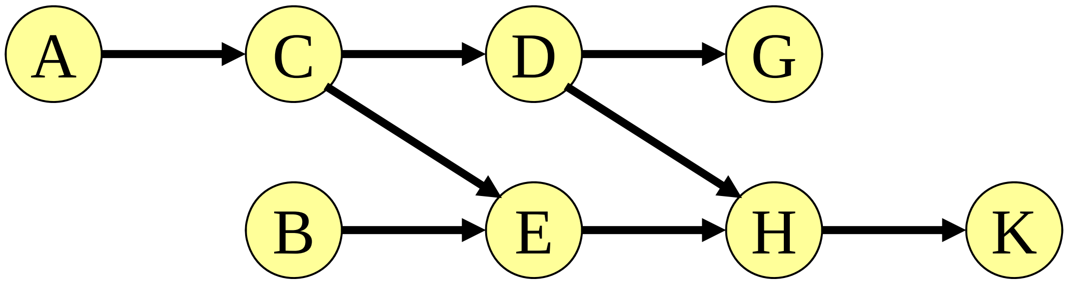
output: G K H

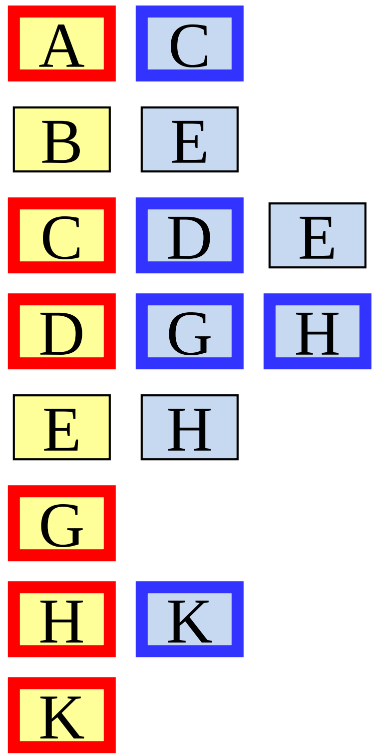
reverse:



Topological Sort example

```
tsort(v)    {
        mark v visited
 D ➔ for each w adjacent to v if w unvisited tsort(w)
        display(v)
        }
```





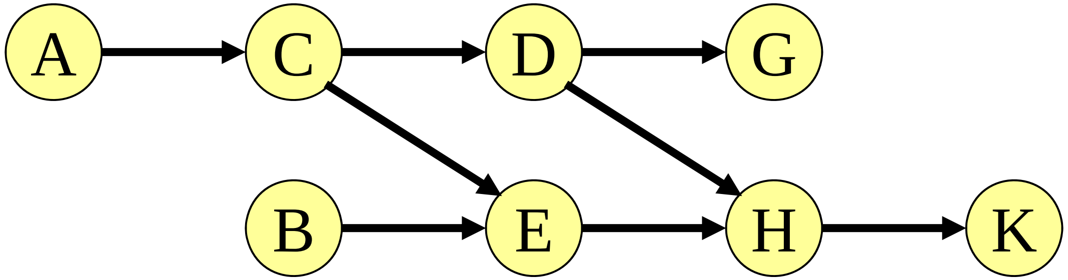
path: A ➔ C ➔ D

output: G K H

reverse:

Topological Sort example

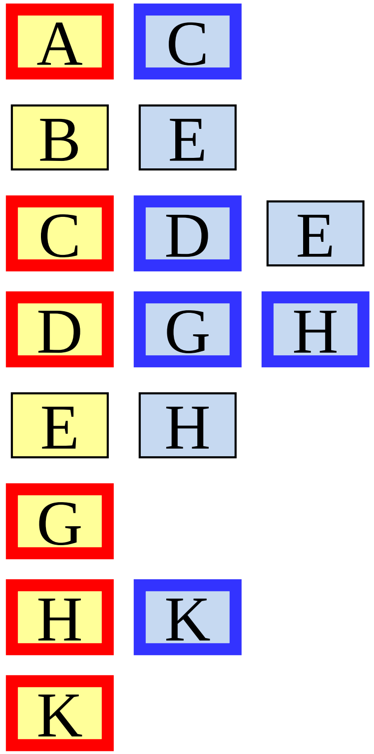
```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 D → display(v)
        }
```



path: **A → C → D**

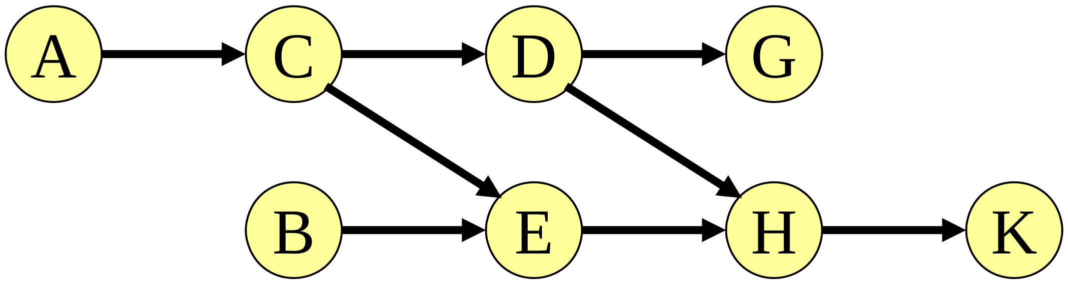
output: **G K H D**

reverse:



Topological Sort example

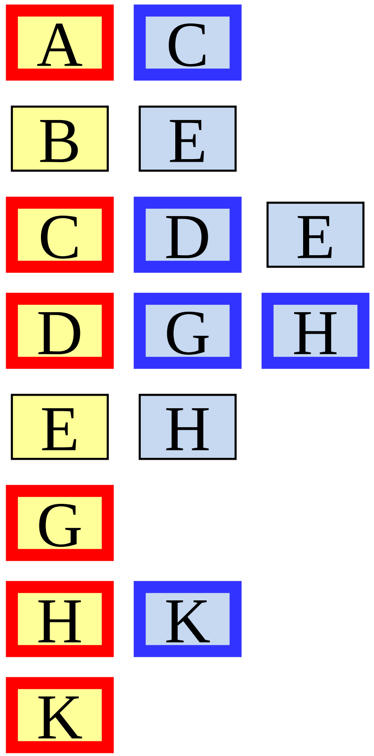
```
tsort(v)    {
      mark v visited
C➔    for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```



path: **A ➔ C**

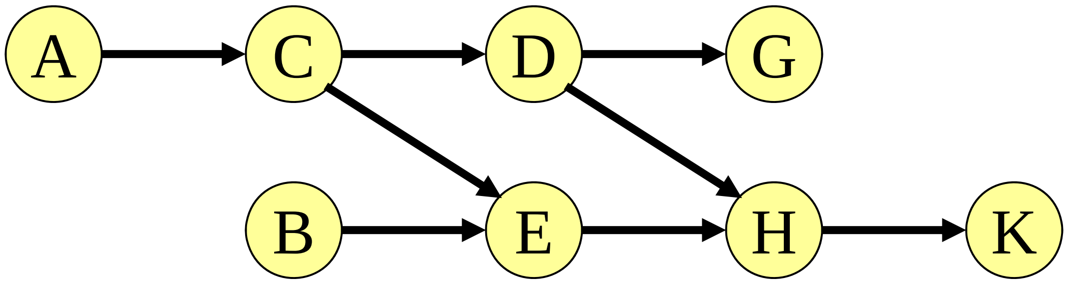
output: **G K H D**

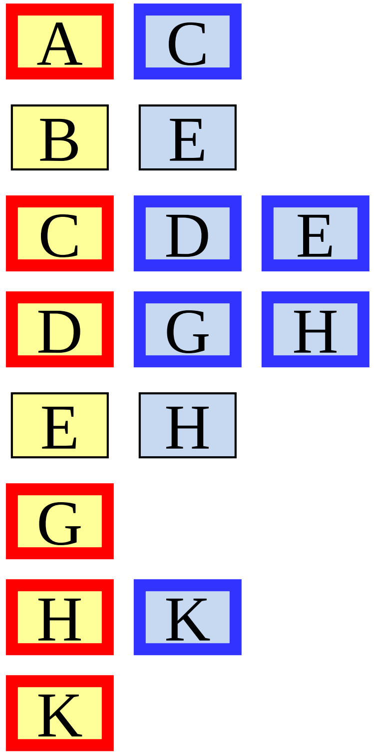
reverse:



Topological Sort example

```
tsort(v)    {
        mark v visited
 C➔     for each w adjacent to v if w unvisited tsort(w)
        display(v)
        }
```





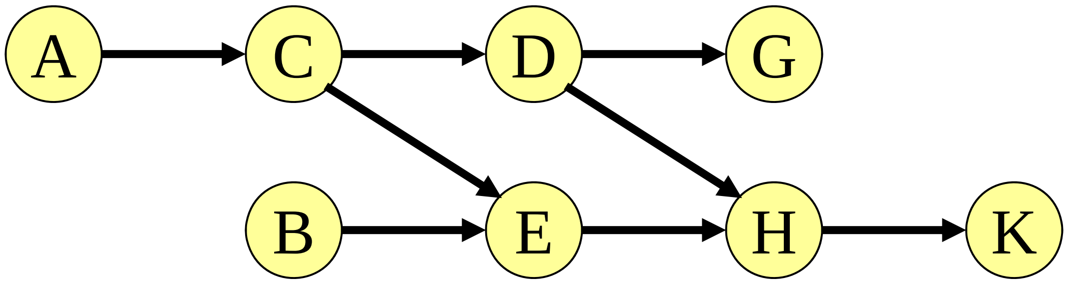
path: A ➔ C ➔ E

output: G K H D

reverse:

Topological Sort example

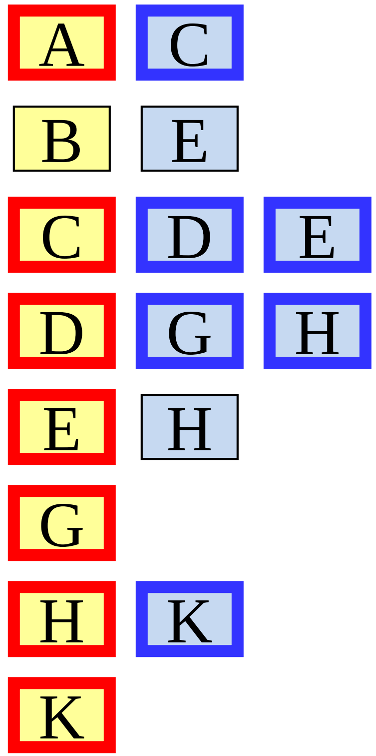
tsort(v) {
E ➔ mark v visited
for each w adjacent to v if w unvisited tsort(w)
display(v)
}



path: A ➔ C ➔ E

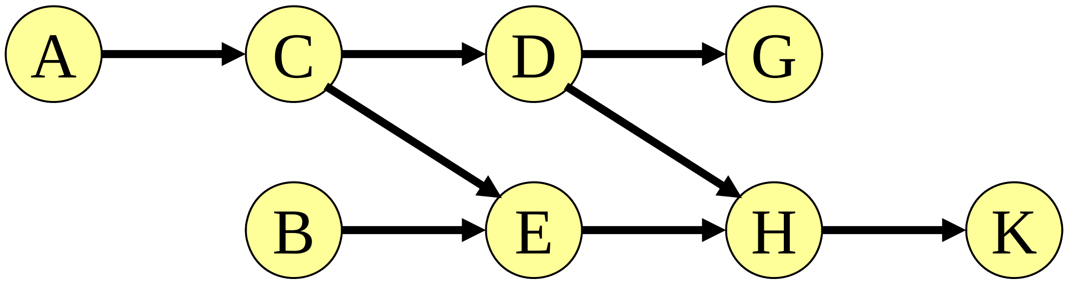
output: G K H D

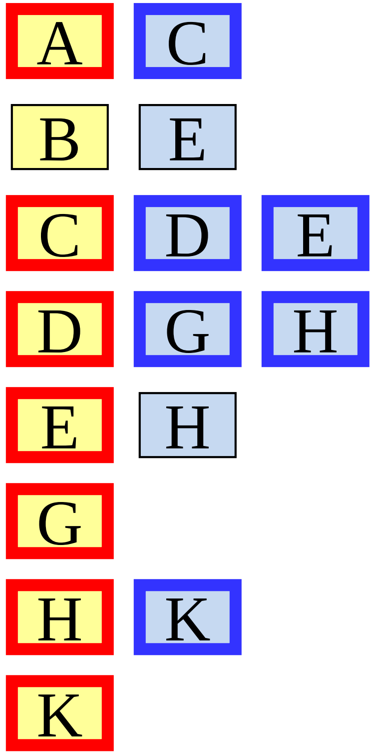
reverse:



Topological Sort example

```
tsort(v)    {
       mark v visited
E ➔    for each w adjacent to v if w unvisited tsort(w)
       display(v)
       }
```





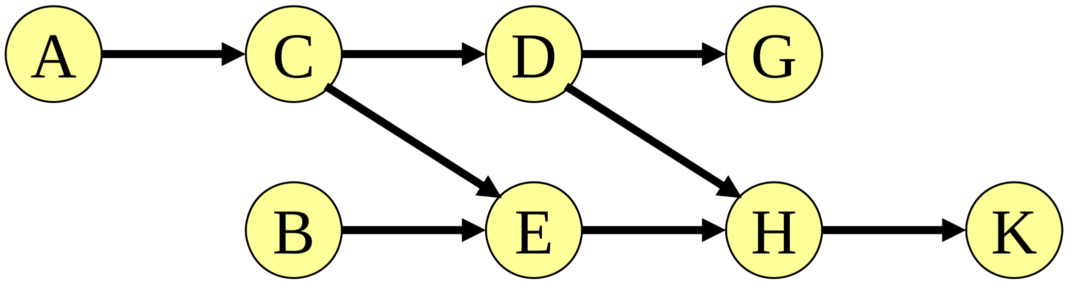
path: **A ➔ C ➔ E**

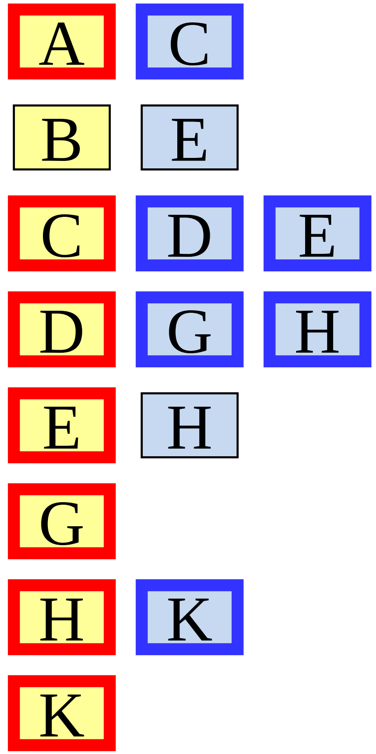
output: **G K H D**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 E → display(v)
        }
```





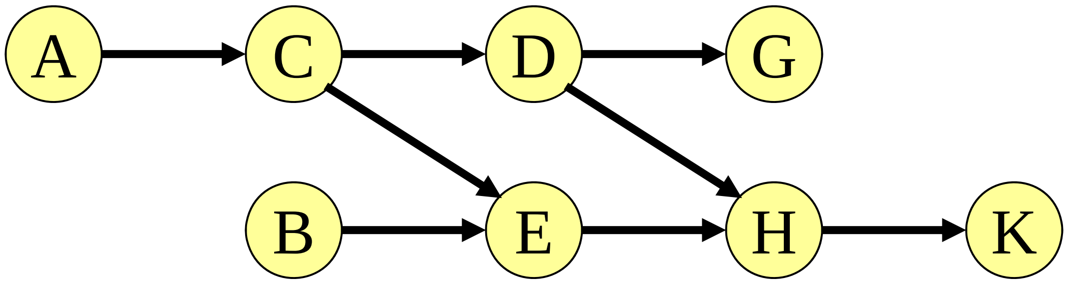
path: **A → C → E**

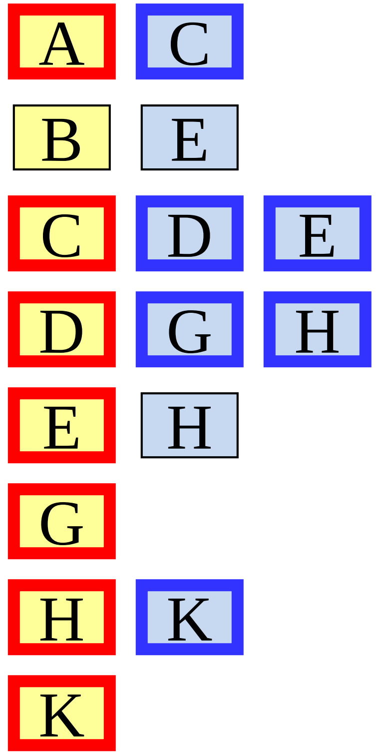
output: **G K H D E**

reverse:

Topological Sort example

```
tsort(v)    {
    mark v visited
C ➔ for each w adjacent to v if w unvisited tsort(w)
    display(v)
    }
```





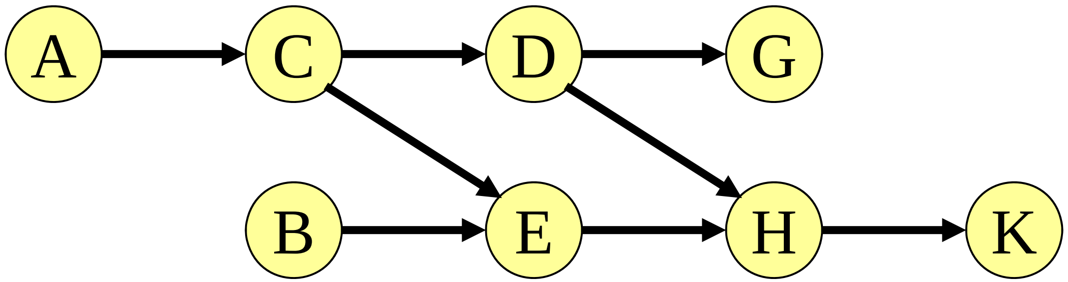
path: **A ➔ C**

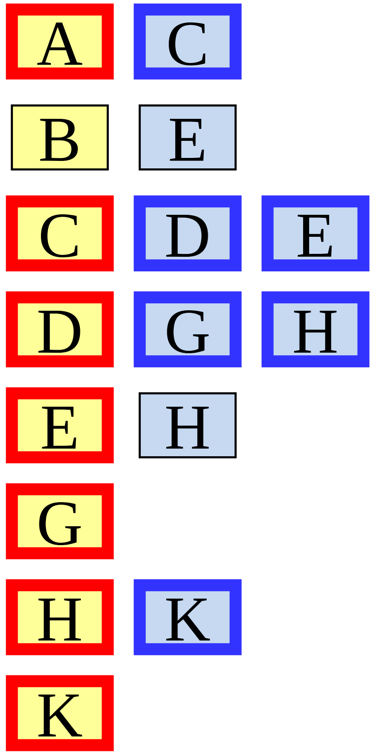
output: **G K H D E**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 C → display(v)
        }
```





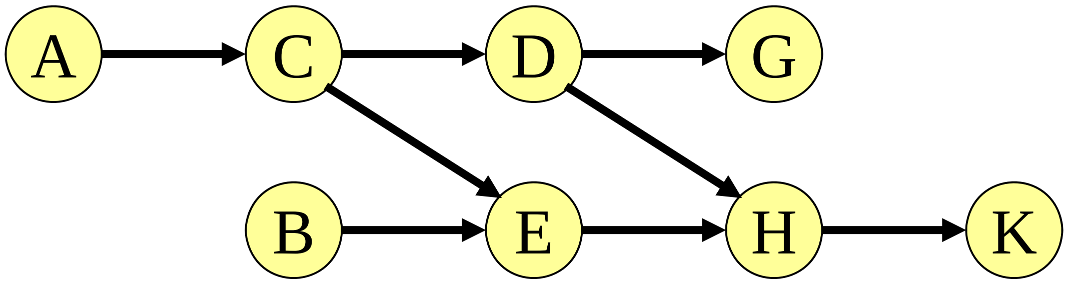
path: **A → C**

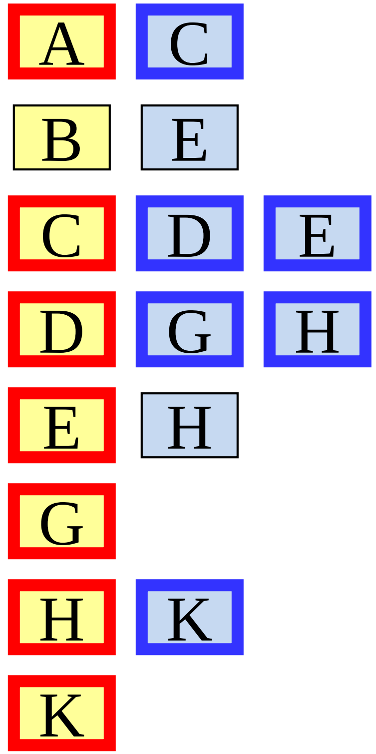
output: **G K H D E C**

reverse:

Topological Sort example

```
tsort(v)    {
      mark v visited
A ➔  for each w adjacent to v if w unvisited tsort(w)
      display(v)
      }
```





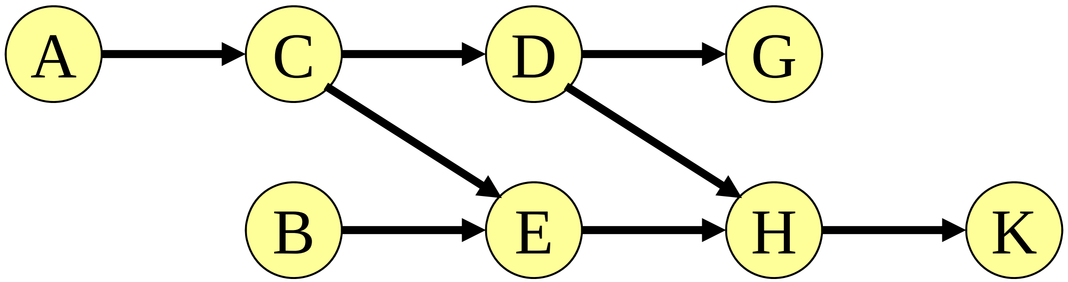
path: **A**

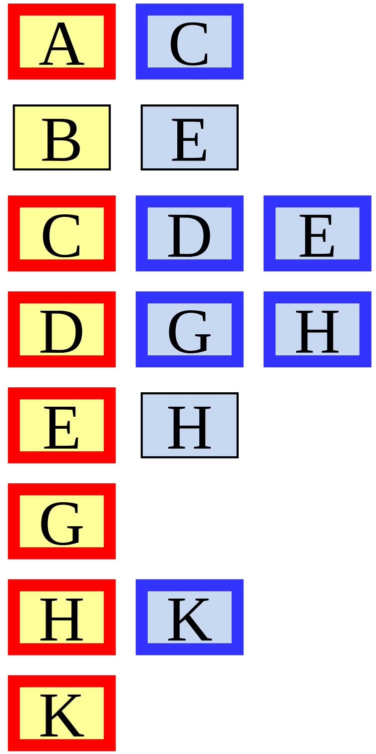
output: **G K H D E C**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 A ➔ display(v)
        }
```





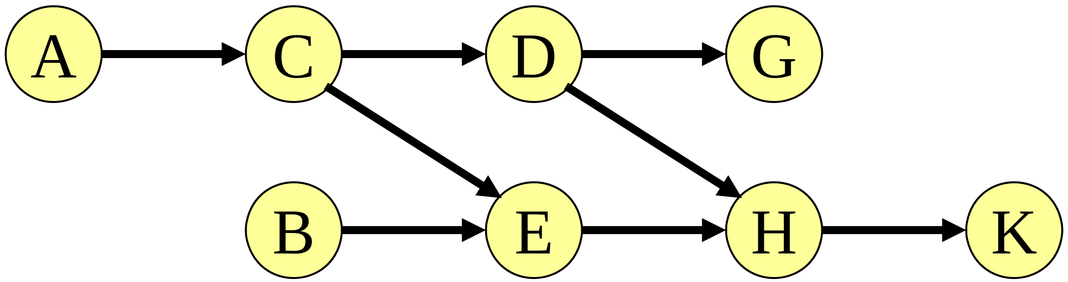
path: **A**

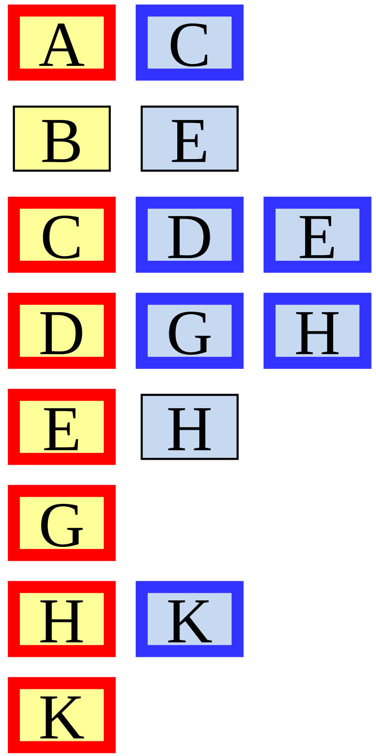
output: **G K H D E C A**

reverse:

Topological Sort example

```
tsort(v)    {
    mark v visited
    for each w adjacent to v if w unvisited tsort(w)
    display(v)
    }
```





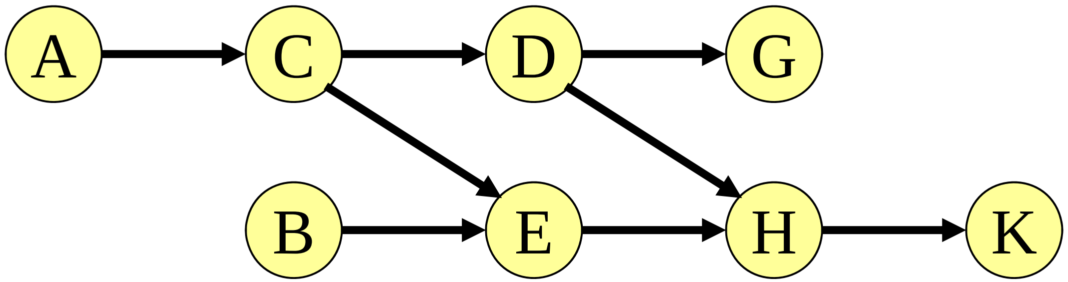
path:

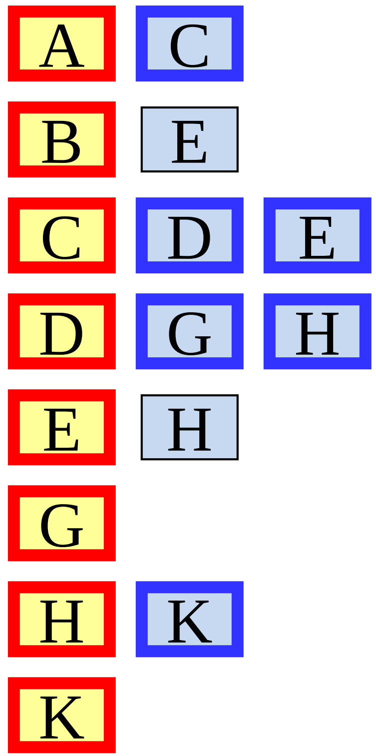
output: **G K H D E C A**

reverse:

Topological Sort example

tsort(v) {
B ➔ mark v visited
for each w adjacent to v if w unvisited tsort(w)
display(v)
}





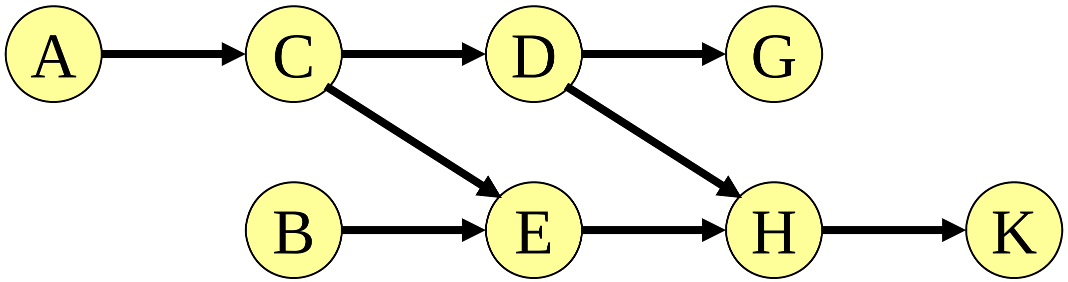
path: **B**

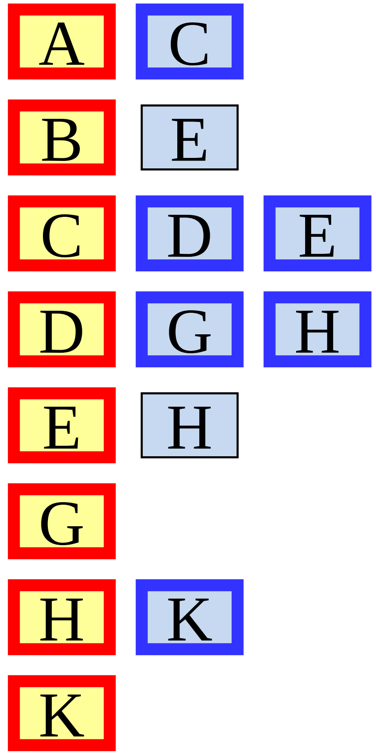
output: **G K H D E C A**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
  B ➔for each w adjacent to v if w unvisited tsort(w)
        display(v)
        }
```





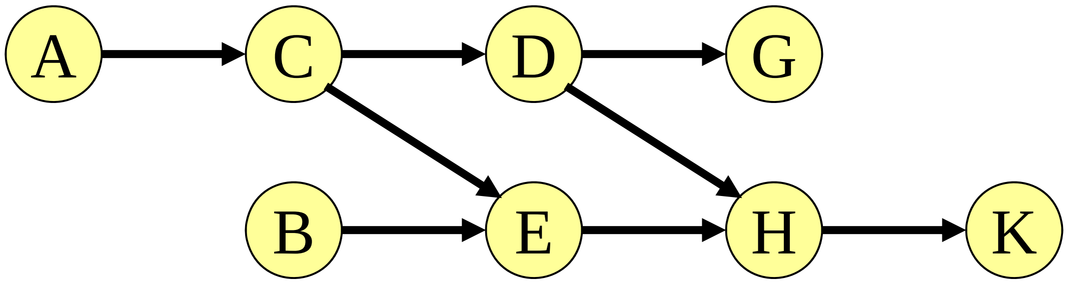
path: **B**

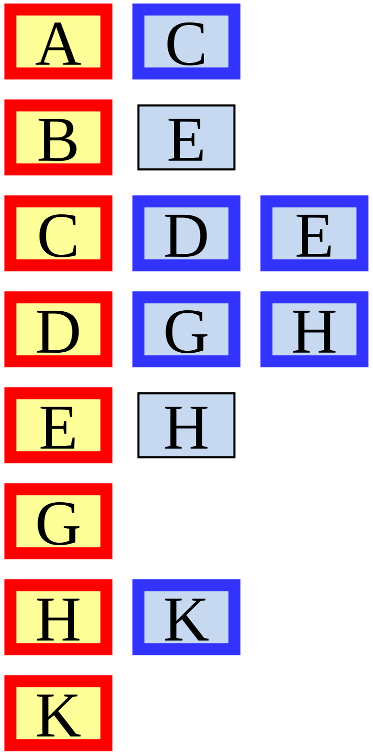
output: **G K H D E C A**

reverse:

Topological Sort example

```
tsort(v)    {
        mark v visited
        for each w adjacent to v if w unvisited tsort(w)
 B ➔ display(v)
        }
```





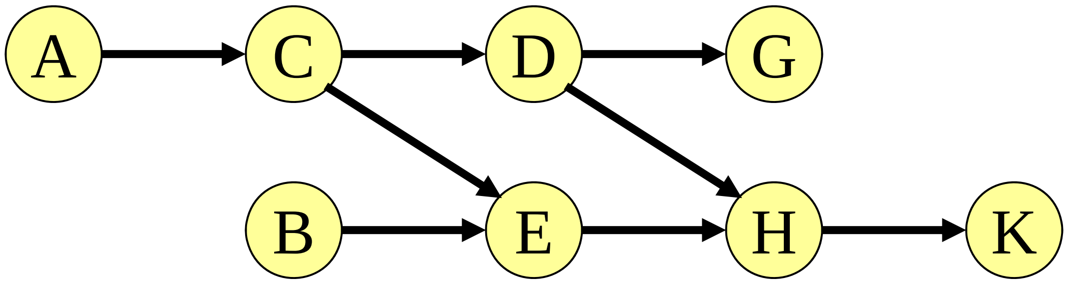
path: **B**

output: **G K H D E C A B**

reverse:

Topological Sort example

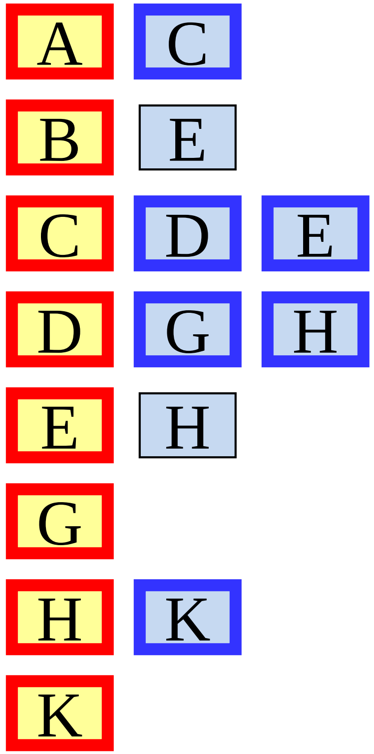
```
tsort(v)    {
    mark v visited
    for each w adjacent to v if w unvisited tsort(w)
    display(v)
    }
```



path:

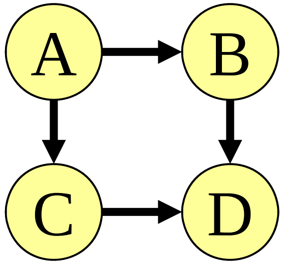
output: **G K H D E C A B**

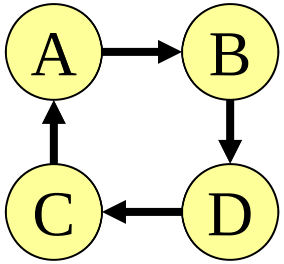
reverse: **B A C E D H K G**



Detecting Cycles

- **Use Warshall**

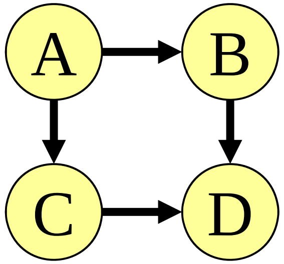


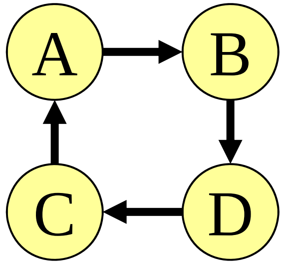


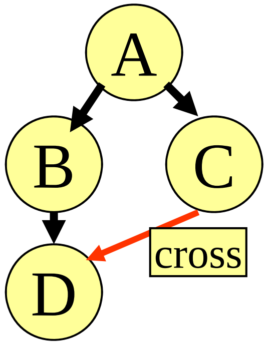
0	1	1	1
0	0	0	1
0	0	0	1
0	0	0	0

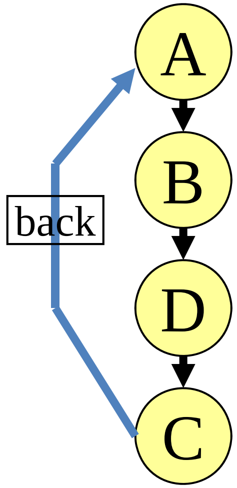
1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

- **Use depth first search**







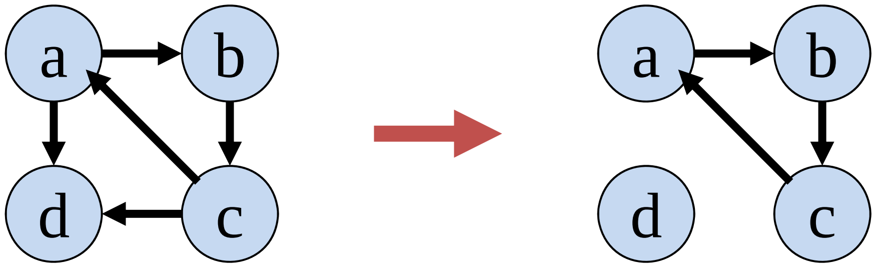


<u>Connectivity</u> - & Reachability (Warshall) Strongly Connected Components (SCCs)

- **Strongly connected component** of a digraph - set of vertices in which there is a **<u>path</u>** from any one vertex in the set to any other vertex in the set
- partition V into equivalence classes V_i , 1 <= i <= r such that v and w are equivalent iff there is a path from v to w and from w to v
- let E_i be the set of edges with head and tail in V_i
- the graphs $G_i = (V_i, E_i)$ are called **STRONGLY CONNECTED COMPONENTS** (**SCC**s) of G
- a **STRONGLY CONNECTED GRAPH** has only one SCC

SCC: example

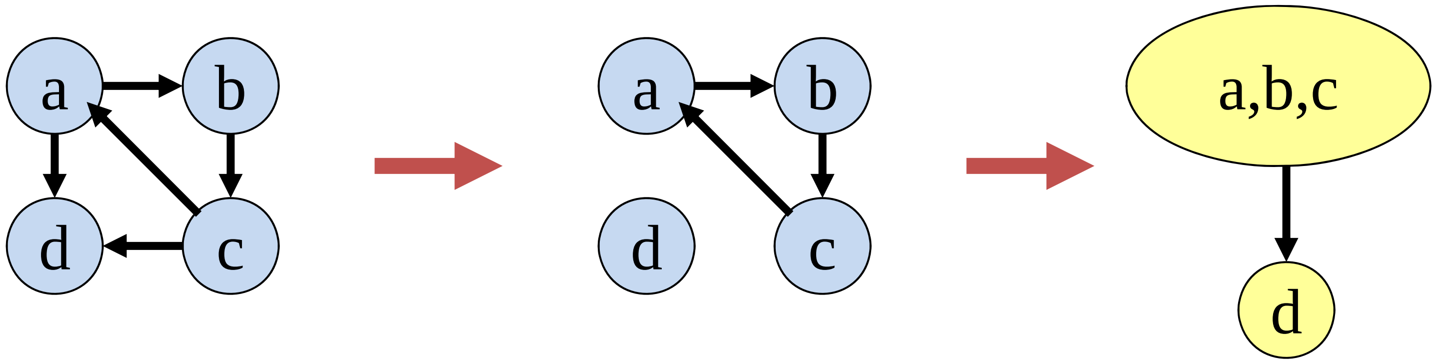
- a digraph and its strongly connected components



- every vertex of G is in some SCC
- **NOT** every edge of G is in some SCC
- **SCC = Strongly Connected Component**

Reduced Graph

- In a reduced graph (RG), the vertices are the **strongly connected components** of G



- edge from vertex C to C' in RG if there is an edge from some vertex in C to some vertex in C'
- RG is always a DAG since if there were a cycle, all components in the cycle would be one strong component

SCCs: algorithm

1. Perform a dfs and assign a number to each vertex

   dfs(v) { mark v visited

   for each w adjacent to v if w unvisited dfs(w)

   number v

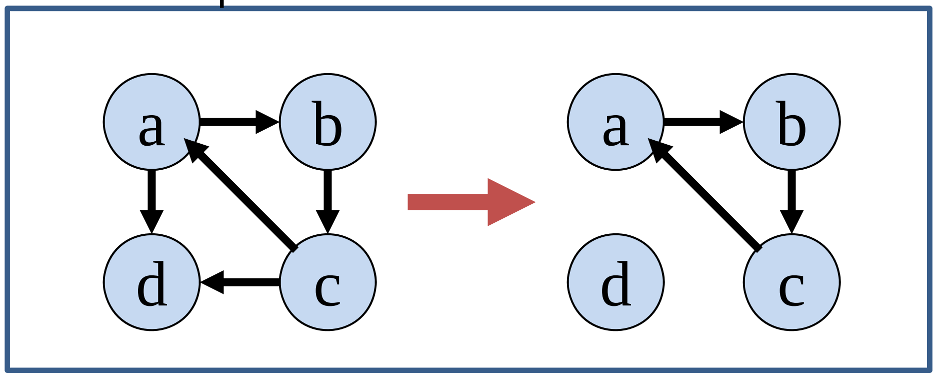
   }

2. construct digraph G_r by reversing every edge in G
3. perform a dfs on G_r starting at highest numbered vertex (repeat on next highest if all vertices not reached)
4. each tree in resulting spanning forest is an SCC of G

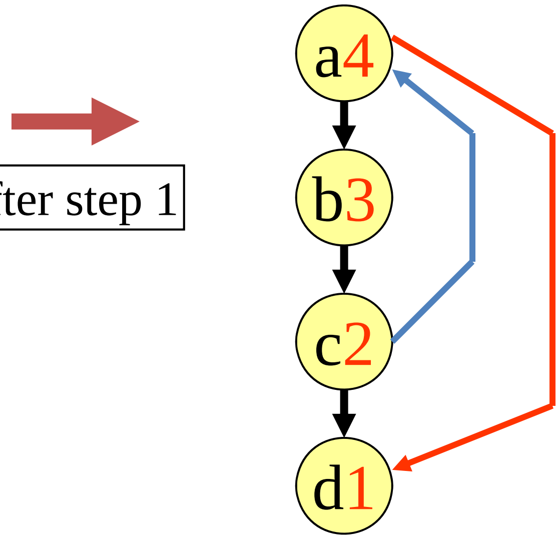
SCCs: example

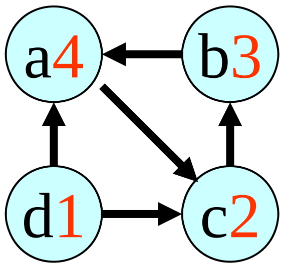
dfs(v) { mark v visited
  for each w adjacent to v if w unvisited dfs(w)
  number v
}

Graph SCCs

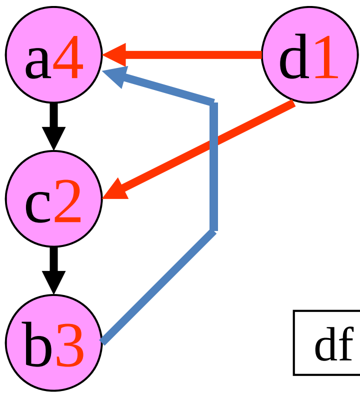


after step 1





Graph G_r



df spanning forest for G_r

Graphs: terminology

- G = (V,E) V = set of vertices, E = set of edges (v,w)
- (v,w) ordered = **digraph** (directed graph)
- (v,w) non-ordered = **undirected graph**
- digraph: w is **adjacent** to v if there is an edge from v to w
- **DAG**: directed acyclic graph
- **path**: sequence of vertices $v_1..v_n$ where $(v_1,v_2)\ldots(v_{n-1},v_n)$ are edges
- **path length**: number of edges in a path
- **simple path**: all vertices are distinct (except possibly the first and last)
- **simple cycle**: simple path, length >=1, begin/end on same vertex

Graphs: terminology

- **Strongly Connected Component**: set of vertices in which there is a path from any vertex in the set to any other vertex in the set
- **Reduced Graph**: vertices are strongly connect components of G
- **Strongly Connected Digraph**: a path from every vertex to every other vertex
- **Complete graph**: if there is an edge between every pair of vertices

- **Implementation**: adjacency matrix or adjacency list

Graphs: algorithms

- **Dijkstra**: single source shortest path
- **Floyd**: all pairs shortest path
- **Warshall**: transitive closure (determines if a path exists from v to w)
- **Depth First Search**:
  - used to derive the depth first spanning forest for the graph
  - used in cycle detection
  - used to derive the strong components
- **Breadth First Search**:
  - used to derive the breadth first spanning forest for the graph
- **Topological Sort**: DAG => sequence