Undirected Graphs: Depth First Search

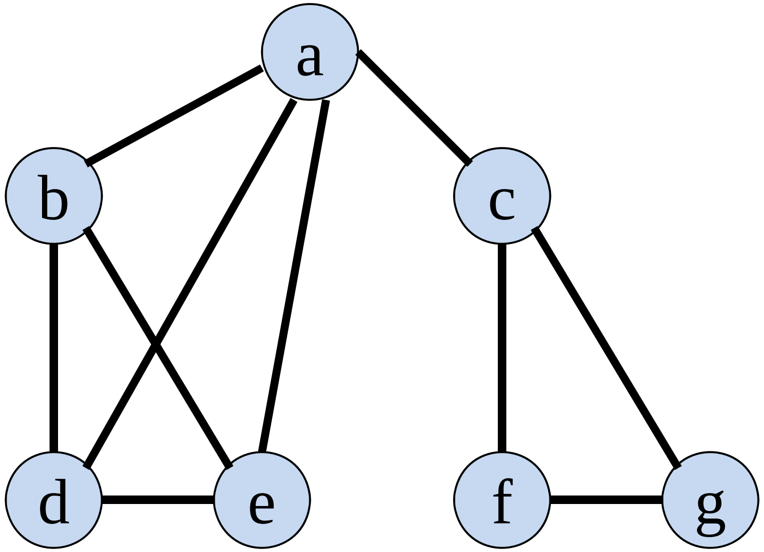
- Similar to the algorithm for directed graphs
- (v, w) is similar to (v,w) (w,v) in a digraph
- for the depth first spanning forest (dfsf), each connected component in the graph will have a tree in the dfsf
  - (if the graph has one component, the dfsf will consist of one tree)
- in the dfsf for **digraphs**, there were 4 kinds of edges: **tree, forward, back** and **cross**
- for a **graph** there are 2: **tree** and **back** edges
  (forward and back edges are not distinguished and there are no cross edges)

Undirected Graphs: Depth First Search

- **Tree edges**:
  - edges (v,w) such that dfs(v) directly calls dfs(w) **(or vice versa)**
- **Back edges**:
  - edges (v,w) such that neither dfs(v) nor dfs(w) call each other directly **(e.g. dfs(w) calls dfs(x) which calls dfs(v) so that w is an ancestor of v)**

- in a dfs, the vertices can be given a dfs number similar to the directed graph case

DFS: Example

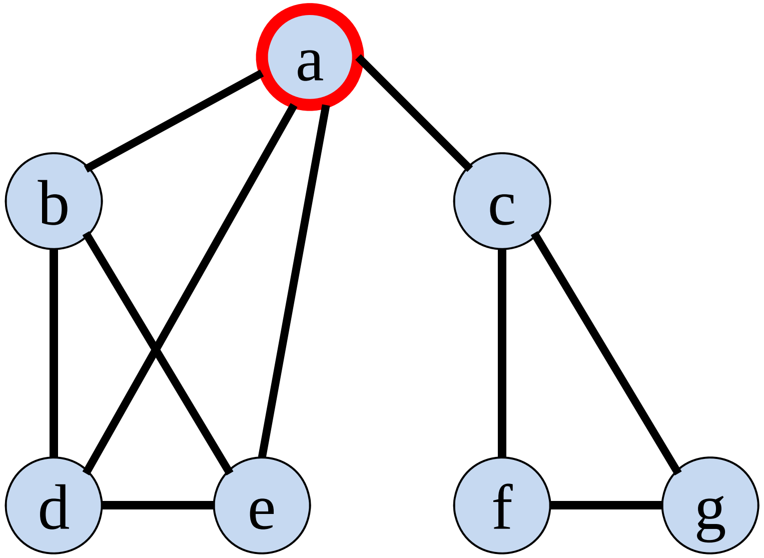


a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

start: a

a b d e c f g

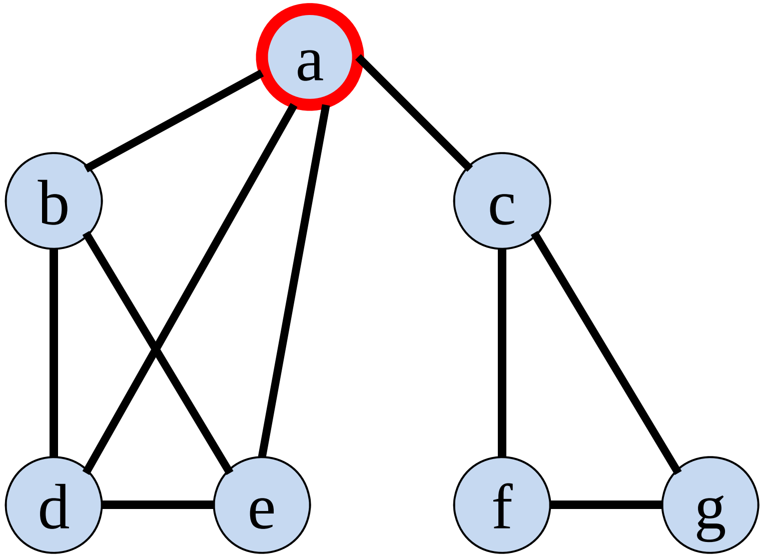
DFS: Example



a(1)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

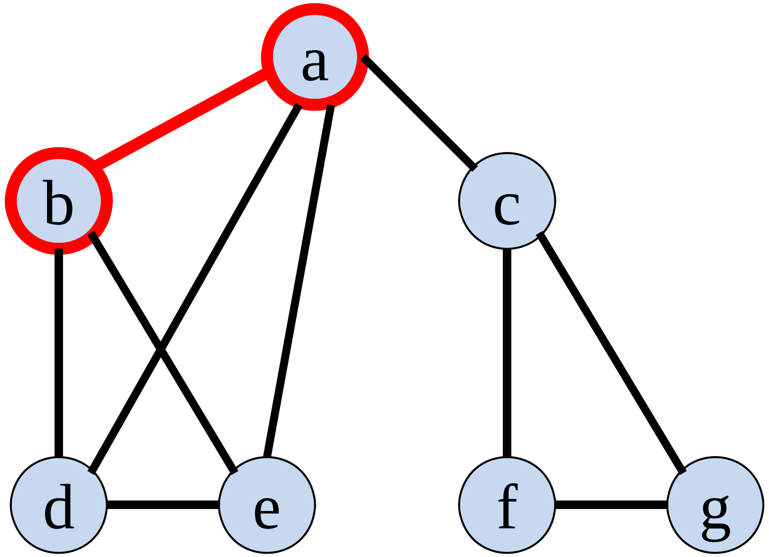
DFS: Example



a(1)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

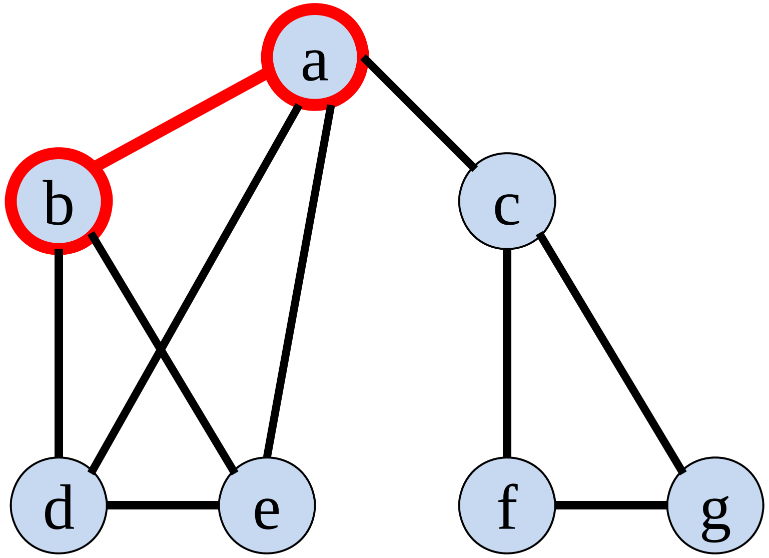
DFS: Example



a(1)→b(2)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

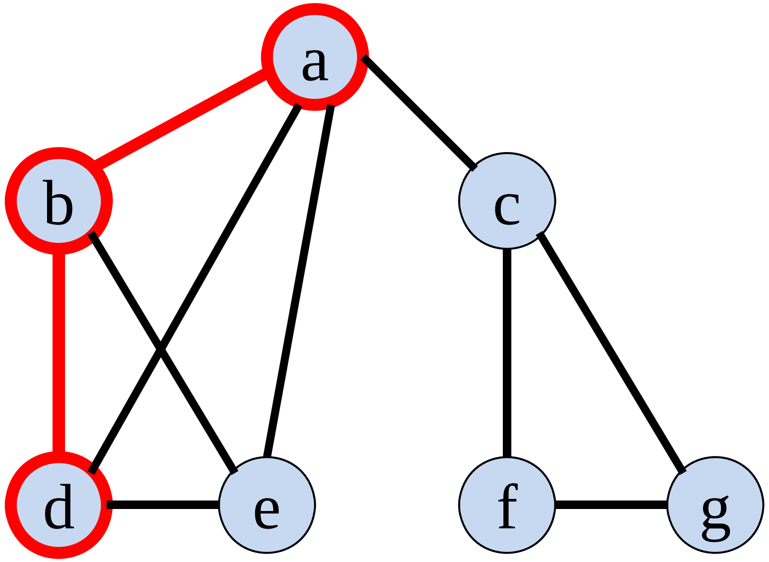
DFS: Example



a(1)➔b(2)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

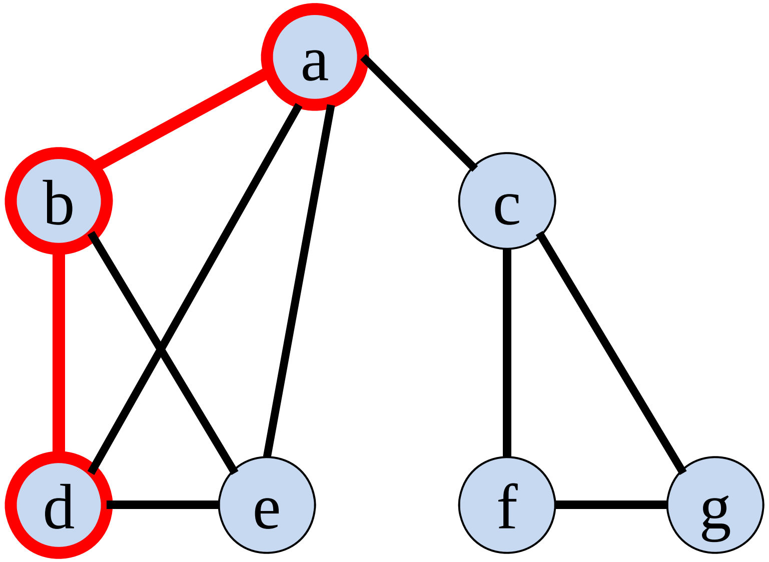
DFS: Example



a(1)➔b(2)➔d(3)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

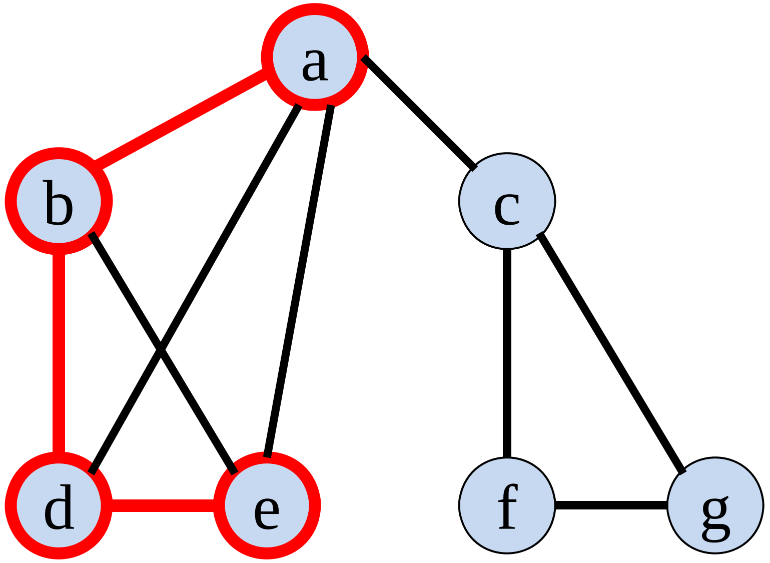
DFS: Example



a(1)→b(2)→d(3)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

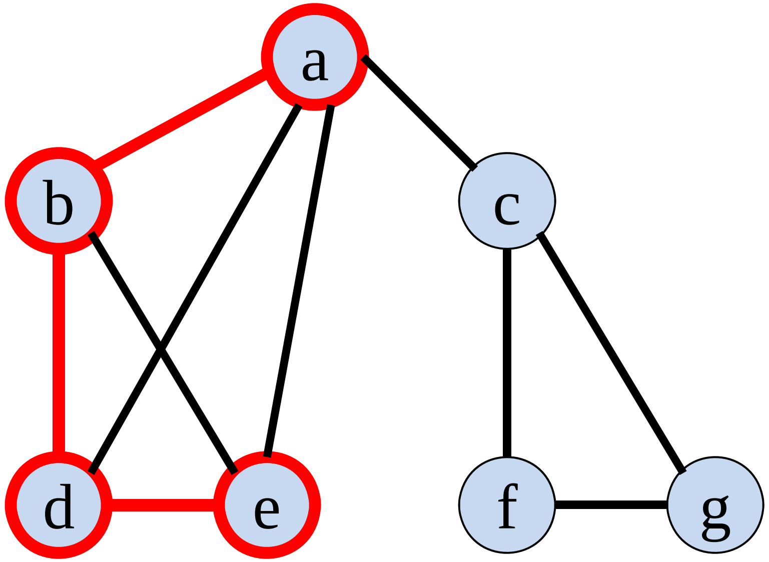
DFS: Example



a(1)➔b(2)➔d(3)➔e(4)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

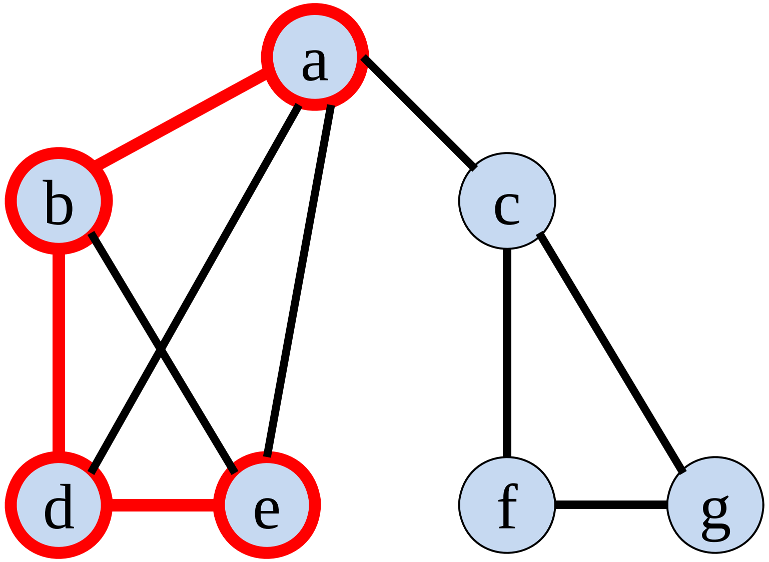
DFS: Example



a(1)➔b(2)➔d(3)➔e(4)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

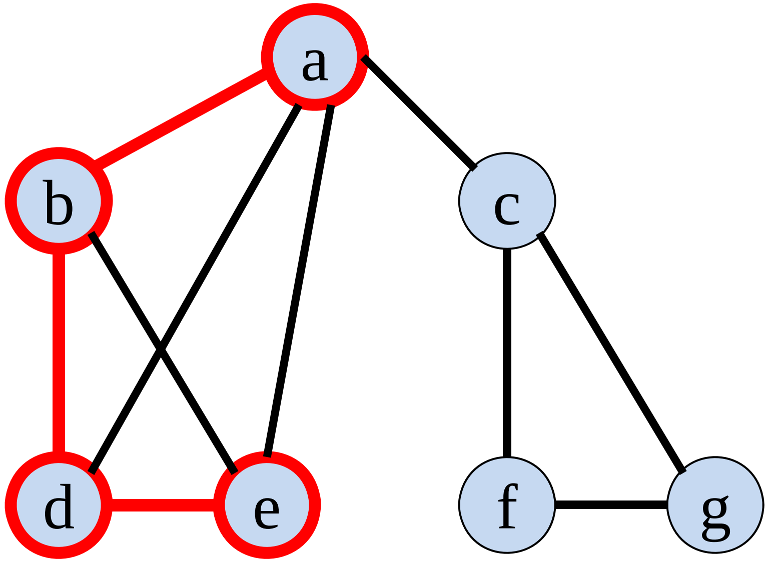
DFS: Example



a(1)➔b(2)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

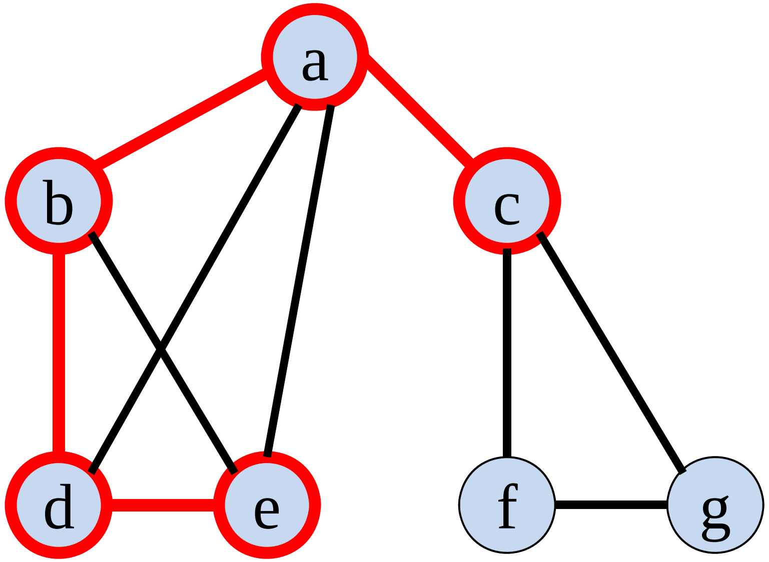
DFS: Example



a(1)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

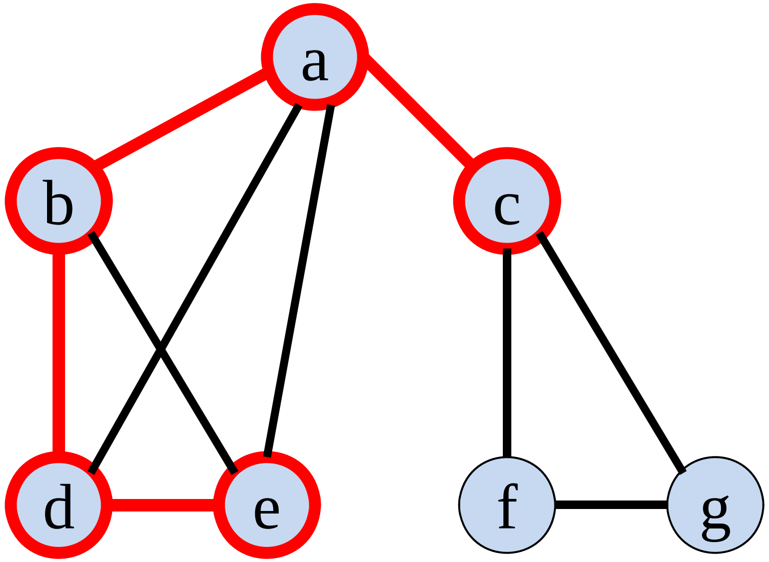
DFS: Example



a(1)→c(5)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

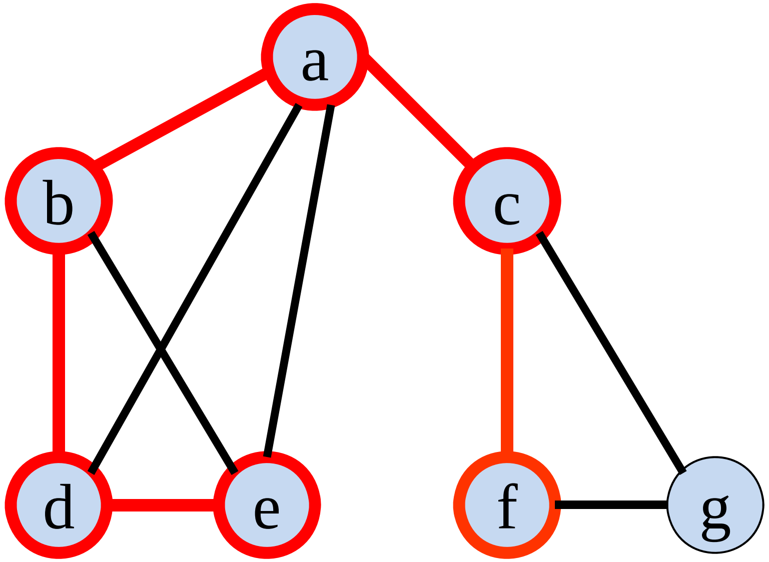
DFS: Example



a(1)→c(5)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

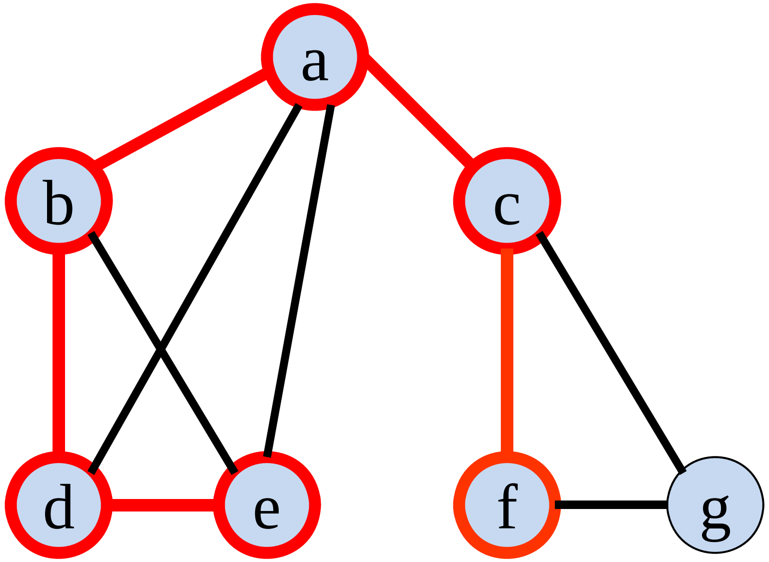
DFS: Example



a(1)➔c(5)➔f(6)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

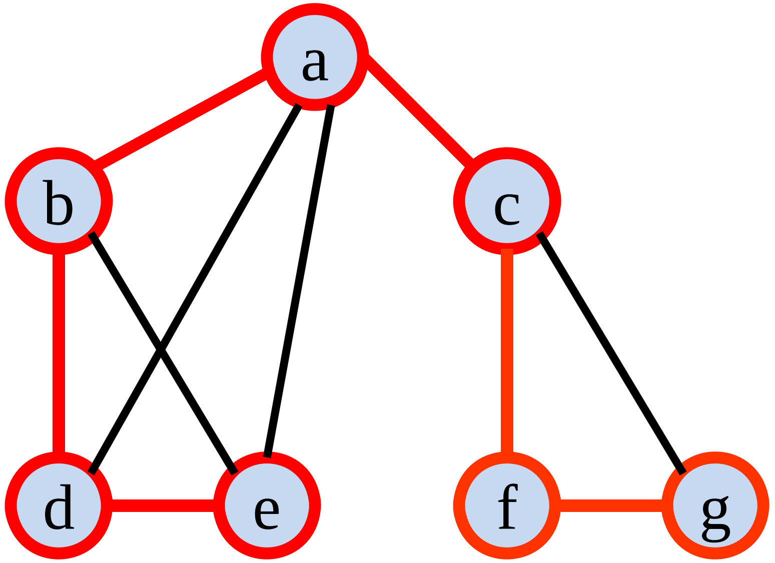
DFS: Example



a(1)➔c(5)➔f(6)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

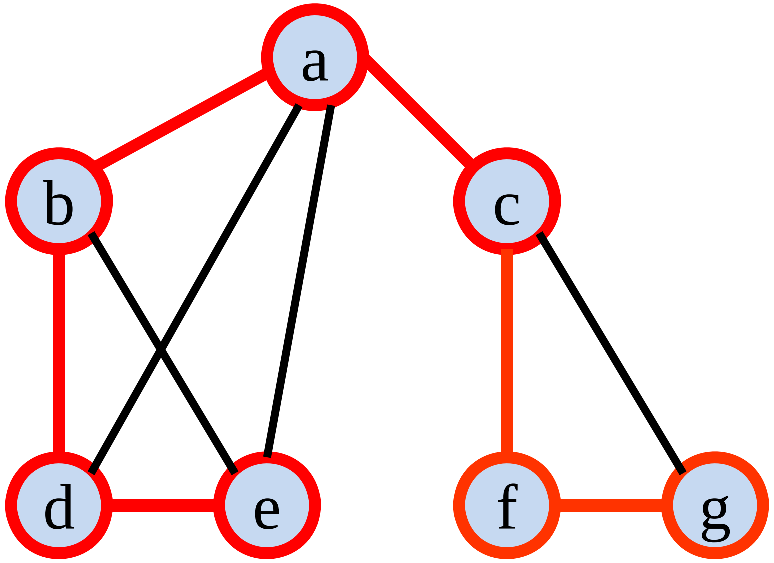
DFS: Example



a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

a(1)➔c(5)➔f(6)➔g(7)

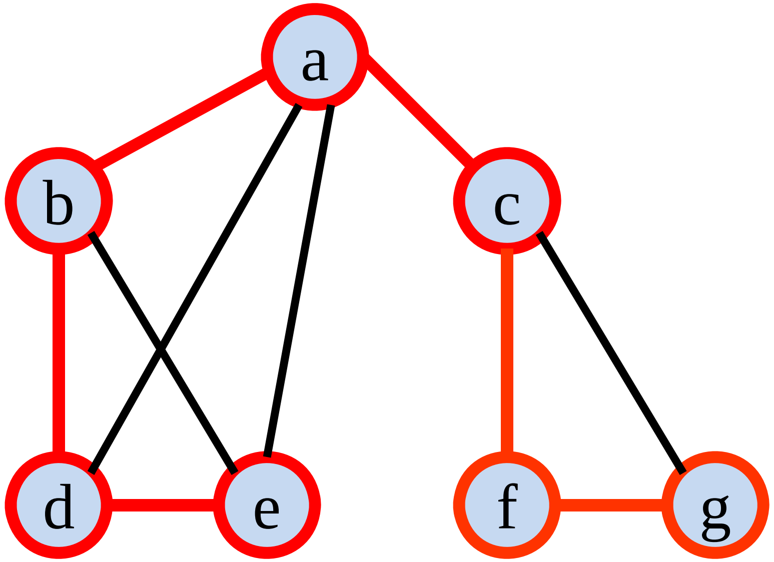
DFS: Example



a(1)➔c(5)➔f(6)➔g(7)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

DFS: Example



a(1)➔c(5)

a b c d e

b a d e

c a f g

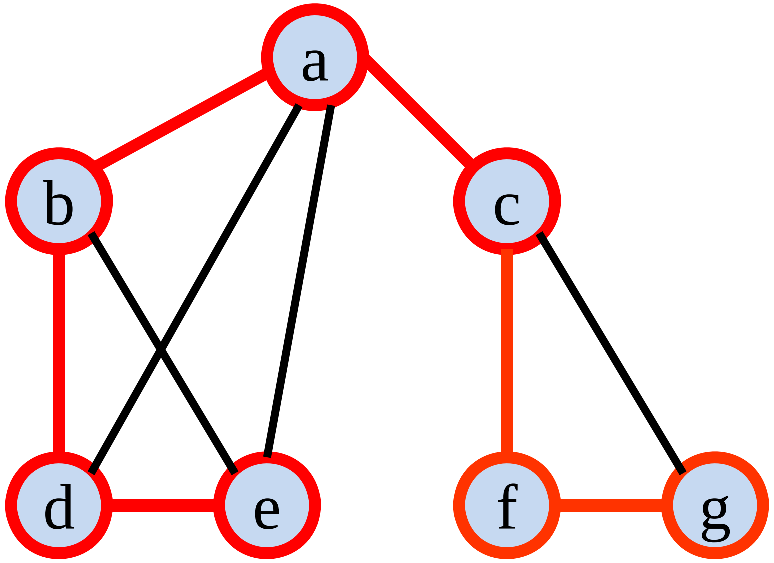
d a b e

e a b d

f c g

g c f

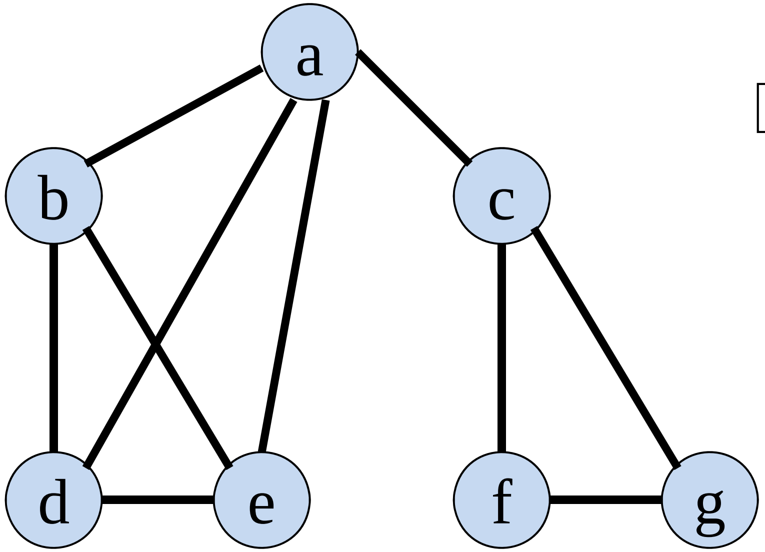
DFS: Example

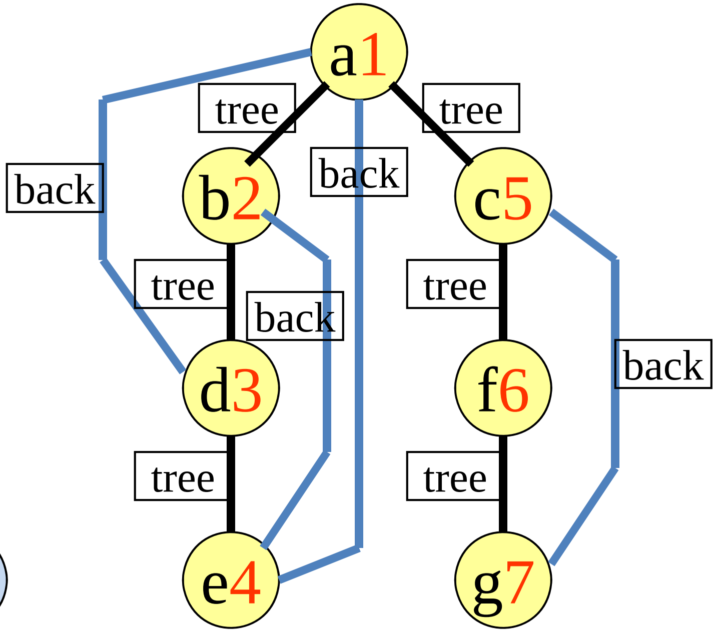


a(1)

a	b	c	d	e
b	a	d	e	
c	a	f	g	
d	a	b	e	
e	a	b	d	
f	c	g		
g	c	f		

DFSF: Example (Depth-First Spanning Forest)

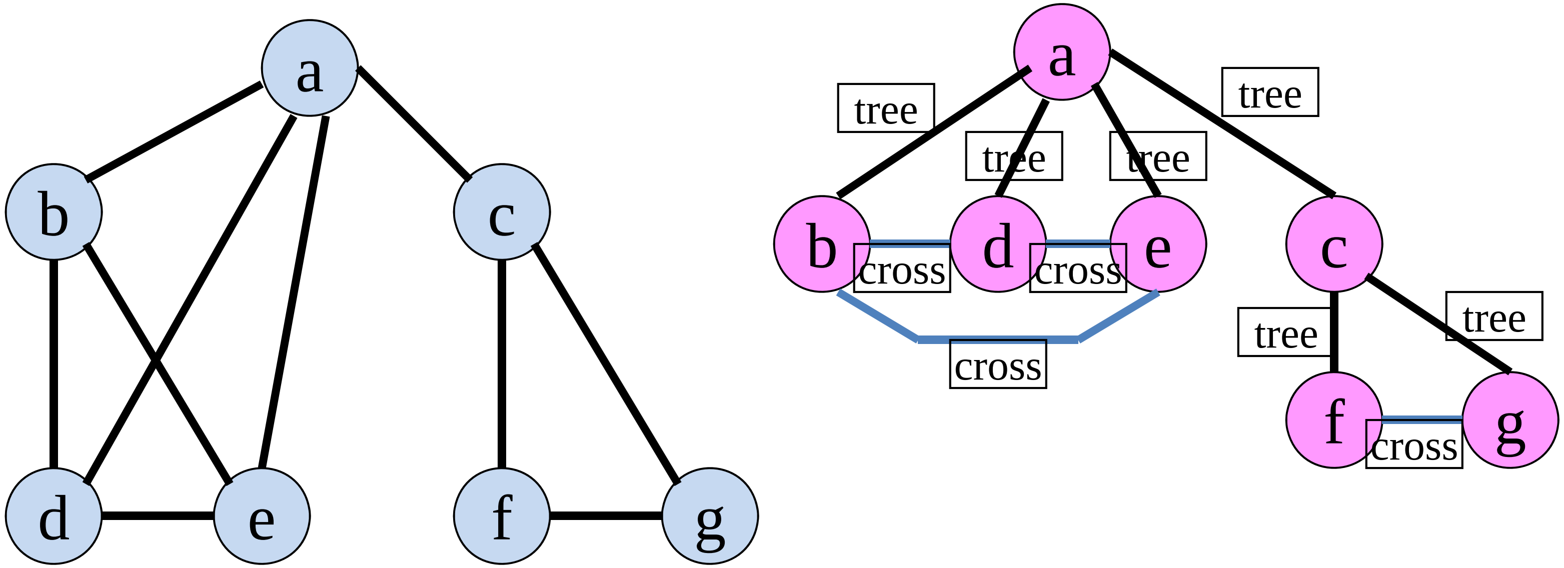




Undirected Graphs: Breadth First Search

- for each vertex v, visit all the adjacent vertices first
- a breadth-first spanning forest can be constructed
  - consists of
    - tree edges: edges (v,w) such that v is an ancestor of w (or vice versa)
    - cross edges: edges which connect two vertices such that neither is an ancestor of the other
- NB the search only works on one connected component
  - if the graph has several connected components then apply bfs to each component

BFSF: Example



Note that this represents the MST for an unweighted undirected graph

BFSF: algorithm (Breadth-First Spanning Forest)

```
bfs ( )
{       mark v visited; enqueue (v);
        while ( not is_empty (Q) ) {
                x = front (Q); dequeue (Q);
                for each y adjacent to x if y unvisited {
                        mark y visited; enqueue (y);
                        insert ( (x, y) in T );
                        }
                }
        }
```

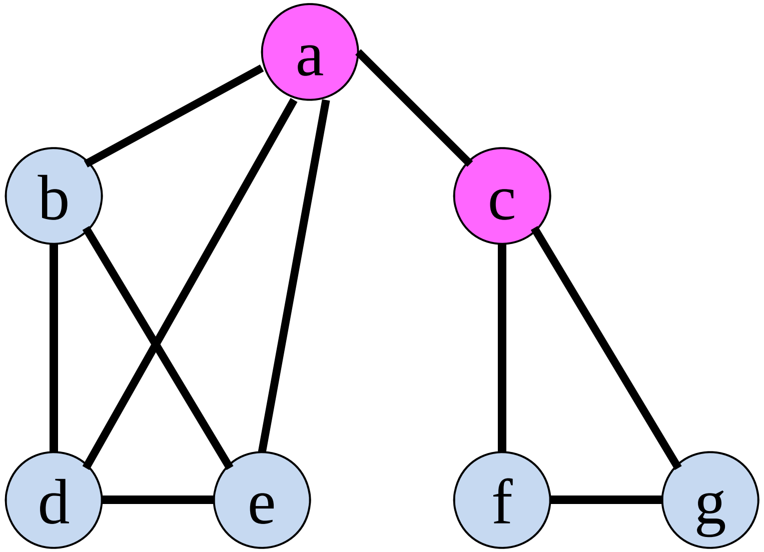
Articulation Point

- An articulation point of a graph is a vertex v such that if v and its incident edges are removed, a **connected component** of the graph is broken into two or more pieces
- a **connected component** with no articulation points is said to be **biconnected**
- the dfs can be used to help find the biconnected components of a graph
- finding articulation points is one problem concerning the **connectivity** of graphs

Connectivity

- finding articulation points is one problem concerning the connectivity of graphs
- a graph has **connectivity k** if the deletion of any (k-1) vertices fails to disconnect the graph (what does this mean?)
  - e.g. a graph has connectivity 2 or more iff it has no articulation points i.e. iff it is biconnected
- the higher the connectivity of a graph, the more likely the graph is to survive failure of some of its vertices
  - e.g. a graph representing sites which must be kept in communication (computers / military / other)

Articulation Points / Connectivity: Example



- articulation points are **a** and **c**
- removing **a** gives {b,d,e} and {c,f,g}
- removing **c** gives {a,b,d,e} and {f,g}
- removing any other vertex does not split the graph

Articulation Points: Algorithm

- Perform a **dfs of the graph**, computing the df-number for each vertex v

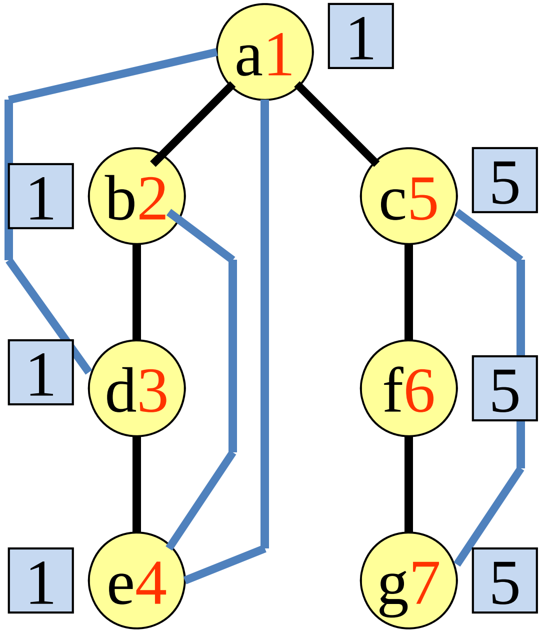
  (df-numbers order the vertices as in a pre-order traversal of a tree)

- for each vertex v, compute low(v) - the smallest df-number of v or any vertex w reachable from v by following down 0 or more tree edges to a descendant x of v (x may be v) and then following a back edge (x, w)
- compute **low(v)** for each vertex v by visiting the vertices in post-order traversal
- when v is processed, **low(y)** has already been computed for all children y of v

Articulation Points: Algorithm

- low(v) is taken to be the **minimum** of
  - df-number(v)
  - df-number(z) for any vertex z where (v,z) is a back edge
  - low(y) for any child y of v
- example
  - **e = min(4, (1,2), -)**
  - **d = min(3, 1, 1) b = min(2, -, 1)**
  - **g = min(7, 5, -) f = min(6, -, 5)**
  - **c = min(5, -, 5)**
  - **a = min(1, -, (1,5))**

- example



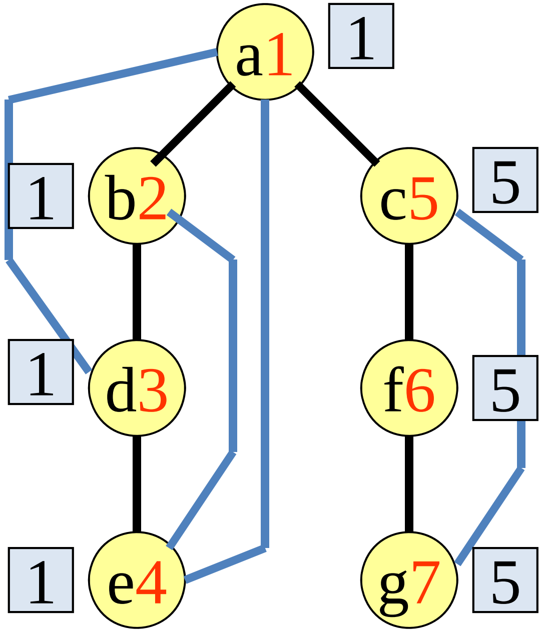
Articulation Points: Algorithm

- the root is an AP iff it has 2 or more children
  - since it has no **cross edges**, removal of the root must disconnect the sub-trees rooted at its children
  - removing a => {b, d, e} and {c, f, g}
- a vertex v **(other than the root)** is an AP iff there is some child w of v such that low(w) >= df-number(v)
  - v disconnects w and its descendants from the rest of the graph
  - if **low(w) < df-number(v)** there must be a way to get from w down the tree and back to a proper ancestor of v (the vertex whose df-number is low(w)) and therefore deletion of v does not disconnect w or its descendants from the rest of the graph

Articulation Points: Example 1

- root - 2 or more children
- other vertices
  - some child w of v such that low(w) >= df-number(v)
- example

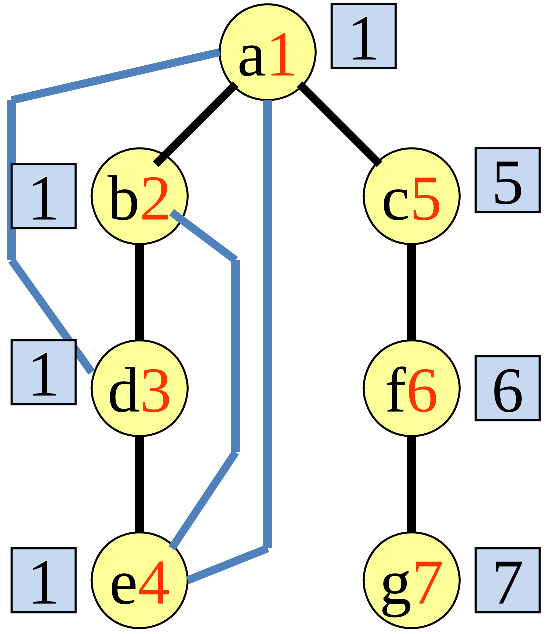
[✓ a]	root >= 2 children
b	low(e) = 1 dfn = 2
[✓ c]	low(g) = 5 dfn = 5
d	low(e) = 1 dfn = 3
e	N/A
f	low(g) = 5 dfn = 6
g	N/A



Articulation Points: Example 2

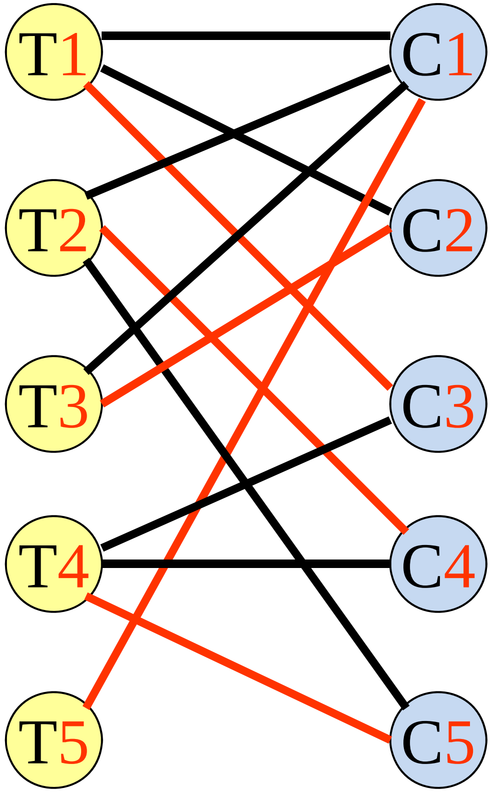
- root - 2 or more children
- other vertices
  - some child w of v such that low(w) >= df-number(v)
- example

[✓ a]	root >= 2 children
b	low(e) = 1 dfn = 2
[✓ c]	low(g) = 7 dfn = 5
d	low (e) = 1 dfn = 3
e	N/A
[✓ f]	low(g) = 7 dfn = 6
g	N/A



Bipartite Graph

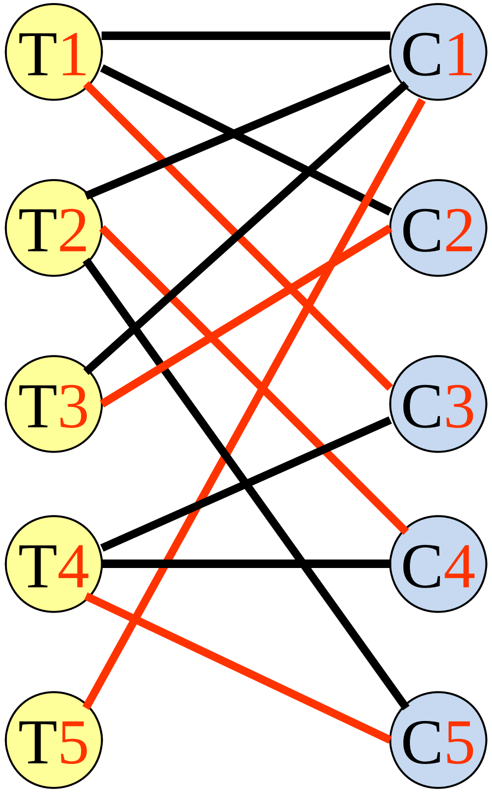
- A graph G is **bipartite** if V is the disjoint union of V_1 and V_2 such that no x_i and x_j in V_1 are adjacent (similarly y_i and y_j in V_2)
- example
  - set of courses
  - set of teachers
  - edge => can teach course
  - **(marriage problem!)**

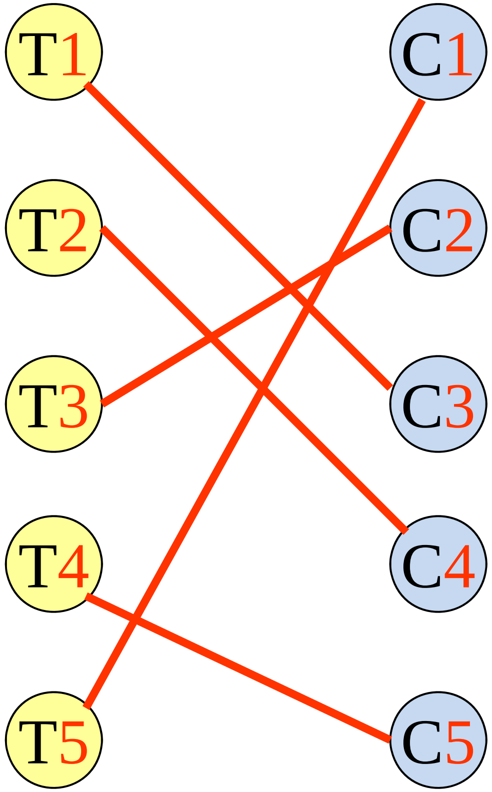


Bipartite Graph: Matching Problem

- A matching in a bipartite graph (BG) is a set of edges whose end points are distinct
- a matching is **complete** if every member of V_1 is the end point of one of the edges in the matching
- a matching is **perfect** if every member of V is the end point of one of the edges in the matching
- in a BG where $V = V_1$ disjoint union V_2, there is a **complete matching** iff for every subset C of V_1 there are at least |C| vertices in V_2 adjacent to members of C
- in a BG where $V = V_1$ disjoint union V_2, there is a **perfect matching** iff for every subset C of V_1 there are at least |C| vertices in V_2 adjacent to members of C and $| V_1 | = | V_2 |$

BG Matching: Example



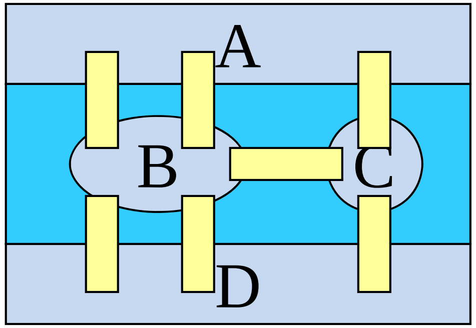


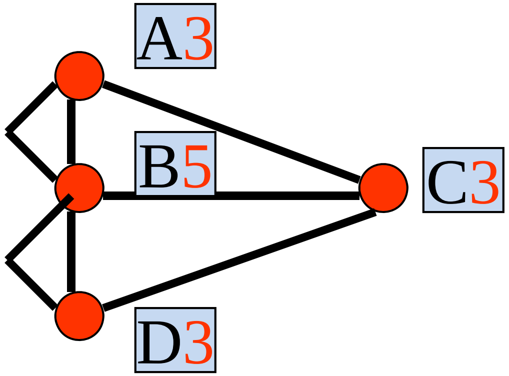
Königsberg Bridge Problem (Euler)

- Find a cycle in the graph G that includes all the vertices and all the edges in G –

  Euler Cycle

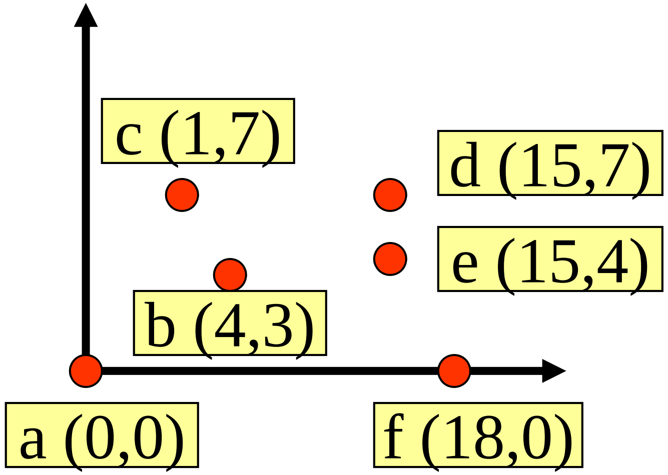
- if G has an Euler cycle, then G is connected and every vertex has an **even degree**
- **degree(v)** = number of edges incident on v

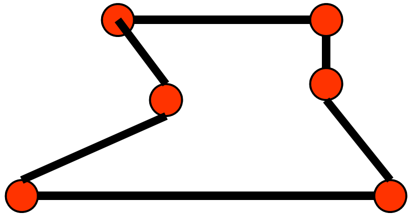




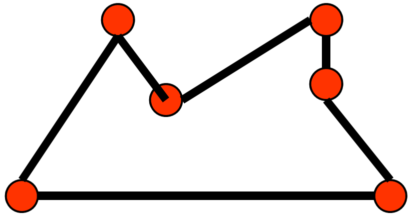
Hamiltonian Cycle

- **Hamiltonian cycle**: cycle in a graph G = (V,E) which contains each vertex in V exactly once, except for the starting and ending vertex that appears twice
- **degree(v) = 2 for all v in V**

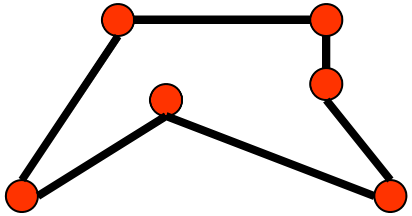




length = 50.0



length = 49.78

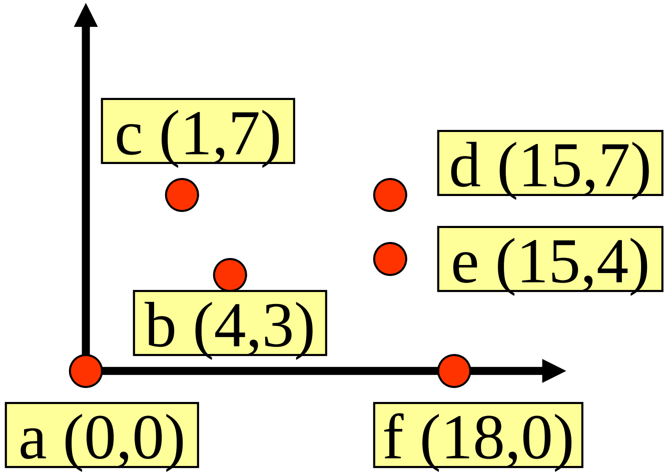


length = 48.39

TSP Problem

- <u>**What may we assume?**</u>
- Graph is fully connected

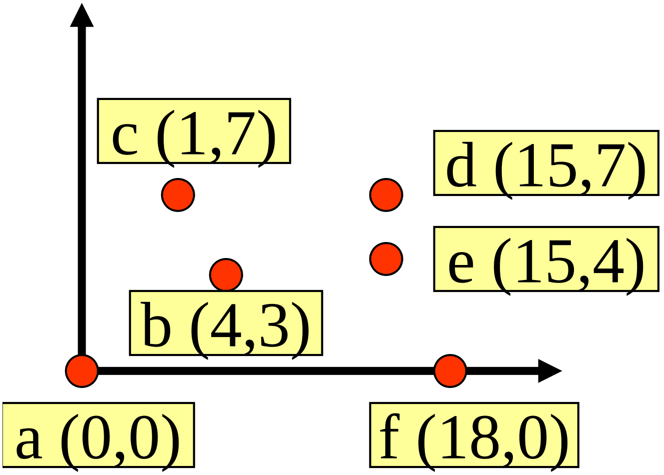
a-b,5	= 5
a-c,sqrt(50)	= 7+
a-d,sqrt(274)	= 16+
a-e,sqrt(241)	= 15+
a-f,18	= 18
b-c,5	= 5
b-d,sqrt(137)	= 11+
b-e,sqrt(122)	= 11+



TSP Problem

Start estimating!

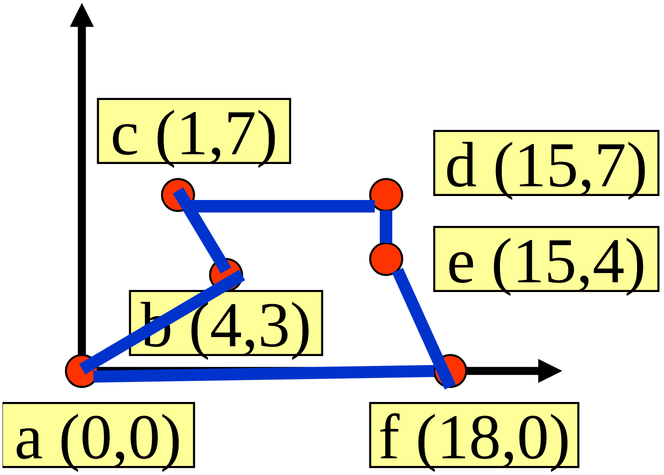
	a	b	c	d	e	f
a		5	7+	16+	15+	18
b	5		5	11+	11+	15+
c	7+	5		14	14+	18+
d	16+	11+	14		3	7+
e	15+	11+	14+	3		5
f	18	15+	18+	7+	5	



TSP Problem

Start estimating!

	a	b	c	d	e	f
a		**5**	7+	16+	15+	**18**
b	**5**		**5**	11+	11+	15+
c	7+	**5**		**14**	14+	18+
d	16+	11+	**14**		**3**	7+
e	15+	11+	14+	**3**		**5**
f	**18**	15+	18+	7+	**5**	

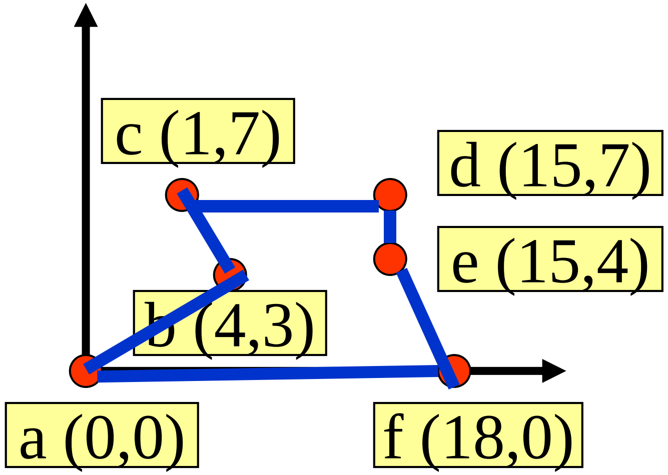


TSP Problem

Adapt Kruskal PQ plus degree max 2 (see below)

1. d-3-e
2. a-5-b, b-5-c, e-5-f
3. c-14-d
4. a-18-f

(0,0,0,1,1,0) ➔ (1,1,0,1,1,0) ➔ (1,**2**,1,1,1,0) ➔ (1,**2**,1,1,**2**,1) ➔ (1,**2**,**2**,**2**,**2**,1) ➔ (**2**,**2**,**2**,**2**,**2**,**2**)



Travelling Salesman Problem (TSP)

- Euler / Hamilton
  - E visits each edge once
  - H visits each vertex once
- to find an Euler cycle - O(n)
- Hamilton
  - factorial or exponential
- Hamilton - applications
  - TSP
  - knight’s tour of n * n board
- TSP
  - Find the minimum-length Hamiltonian cycle for G
  - salesman starts and ends at x
- **TSP Algorithm**
  - **variant of Kruskal’s**
  - **edge acceptance conditions**
    - **degree(v) should not >= 3**
    - **no cycles unless # selected edges = |V|**
    - **greedy / near-optimal**

Graphs: Summary 1

- Directed Graphs
  - G = (V, E)
  - create / destroy G
  - add / remove V
    (=>remove E)
  - add / remove E
  - is_path(v, w)
  - path_length(v, w)
  - is_cycle(v)
  - is_connected(G)
  - is_complete(G)

- Undirected Graphs
  - G = (V, E)
  - create / destroy G
  - add / remove V
    (=>remove E)
  - add / remove E
  - is_path(v, w)
  - path_length(v, w)
  - is_cycle(v)
  - is_connected(G)
  - is_complete(G)

Graphs: Summary 2

- Directed Graphs
  - navigation
    - depth-first search (dfs)
    - breadth-first search (bfs)
    - Warshall
  - spanning forests
    - **df spanning forest (dfsf)**
    - **bf spanning forest (bfsf)**
  - minimum cost algorithms
    - **Dijkstra** **(single path)**
    - **Floyd** **(all paths)**

- Undirected Graphs
  - navigation
    - depth-first search (dfs)
    - breadth-first search (bfs)
    - Warshall
  - spanning forests
    - **df spanning forest (dfsf)**
    - **bf spanning forest (bfsf)**
  - minimum cost algorithms
    - **Prim** **(spanning tree)**
    - **Kruskal** **(spanning tree)**

Graphs: Summary 3

- **Directed Graphs**
  - topological sort (DAG)
  - strong components
  - reduced graph

- **Undirected Graphs**
  - sub-graph
  - induced sub-graph
  - unconnected graph-free tree
  - articulation points
  - connectivity
  - bipartite graph & matching
  - Königsberg Bridge Problem
  - Hamiltonian cycles
  - Travelling Salesman